

## MULTIADAPTIVE GALERKIN METHODS FOR ODES III: A PRIORI ERROR ESTIMATES\*

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**Abstract.** The multiadaptive continuous/discontinuous Galerkin methods  $\text{mcG}(q)$  and  $\text{mdG}(q)$  for the numerical solution of initial value problems for ordinary differential equations are based on piecewise polynomial approximation of degree  $q$  on partitions in time with time steps which may vary for different components of the computed solution. In this paper, we prove general order a priori error estimates for the  $\text{mcG}(q)$  and  $\text{mdG}(q)$  methods. To prove the error estimates, we represent the error in terms of a discrete dual solution and the residual of an interpolant of the exact solution. The estimates then follow from interpolation estimates, together with stability estimates for the discrete dual solution.

**Key words.** multiadaptivity, individual time steps, local time steps, ODE, continuous Galerkin, discontinuous Galerkin,  $\text{mcG}(q)$ ,  $\text{mdG}(q)$ , a priori error estimates, existence, stability, Peano kernel theorem, interpolation estimates, piecewise smooth

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**1. Introduction.** This is part 3 in a sequence of papers [32, 33] on multiadaptive Galerkin methods,  $\text{mcG}(q)$  and  $\text{mdG}(q)$ , for approximate (numerical) solution of ODEs of the form

$$(1.1) \quad \begin{aligned} \dot{u}(t) &= f(u(t), t), \quad t \in (0, T], \\ u(0) &= u_0, \end{aligned}$$

where  $u : [0, T] \rightarrow \mathbb{R}^N$  is the solution to be computed,  $u_0 \in \mathbb{R}^N$  a given initial condition,  $T > 0$  a given final time, and  $f : \mathbb{R}^N \times (0, T] \rightarrow \mathbb{R}^N$  a given function that is Lipschitz-continuous in  $u$  and bounded.

In the previous two parts of our series on multiadaptive Galerkin methods, we proved a posteriori error estimates, through which the time steps are adaptively determined from residual feedback and stability information, obtained by solving a dual linearized problem. In this paper, we prove a priori error estimates for  $\text{mcG}(q)$  and  $\text{mdG}(q)$ . We also prove the stability estimates and interpolation estimates which are essential to the a priori error analysis.

Standard methods for the time-discretization of (1.1) require that the resolution is equal for all components  $U_i(t)$  of the computed approximate solution  $U(t)$  of (1.1). This includes all standard Galerkin or Runge–Kutta methods; see [9, 4, 23, 24, 41, 2]. Using the same time step sequence  $k = k(t)$  for all components could become very costly if the different components of the solution exhibit multiple time scales of different magnitudes. We therefore propose a new representation of the solution in which the difference in time scales is reflected in the *componentwise* time-discretization of (1.1), that is, each component  $U_i(t)$  is computed using an individual time step sequence  $k_i = k_i(t)$ .

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The multiadaptive Galerkin methods mcG( $q$ ) and mdG( $q$ ) first presented in [32] are formulated as extensions of the standard continuous and discontinuous Galerkin methods cG( $q$ ) and dG( $q$ ), studied earlier in detail by Hulme [28, 27], Jamet [29], Delfour, Hager, and Trochu [7], Eriksson, Johnson, and Thomée [16, 30, 11, 12, 10, 13, 14, 15, 8], and Estep et al. [17, 18, 19, 21, 20]. As such, the analysis of the mcG( $q$ ) and mdG( $q$ ) methods can be carried out within the existing framework, but the extension to multiadaptive time-stepping leads to some technical challenges, in particular, proving the appropriate interpolation estimates.

Local (multiadaptive) time-stepping has been explored before to some extent for specific applications, including specialized integrators for the  $n$ -body problem [37, 5, 1] and low-order methods for conservation laws [39, 22, 6]. Early attempts at local time-stepping include [25, 26]. Recently, a new class of related methods, known as asynchronous variational integrators (AVI) with local time steps, has been proposed [31].

**1.1. Main results.** The main results of this paper are a priori error estimates for the mcG( $q$ ) and mdG( $q$ ) methods, respectively, of the form

$$(1.2) \quad \|e(T)\|_{l_p} \leq CS(T) \|k^{2q}u^{(2q)}\|_{L_\infty([0,T],l_1)}$$

and

$$(1.3) \quad \|e(T)\|_{l_p} \leq CS(T) \|k^{2q+1}u^{(2q+1)}\|_{L_\infty([0,T],l_1)}$$

for  $p = 2$  or  $p = \infty$ , where  $C$  is an interpolation constant,  $S(T)$  is a (computable) stability factor, and  $k^{2q}u^{(2q)}$  (or  $k^{2q+1}u^{(2q+1)}$ ) combines local time steps  $k_i = k_i(t)$  with derivatives of the exact solution  $u$ . The norm  $L_\infty(I, \|\cdot\|)$  is defined by  $\|v\|_{L_\infty(I, \|\cdot\|)} = \sup_{t \in I} \|v(t)\|$ . These estimates state that the mcG( $q$ ) method is of order  $2q$  and that the mdG( $q$ ) method is of order  $2q + 1$  in the local time step. We refer to section 6.2 for the exact results. It should be noted that superconvergence is obtained only at synchronized time levels, such as the end-point  $t = T$ . For the general nonlinear problem, we obtain exponential estimates for the stability factor  $S(T)$ . In [34], we prove that for a parabolic model problem, the stability factor remains bounded and of unit size, independent of  $T$  (up to a logarithmic factor).

**1.2. Notation.** The following notation is used throughout this paper. Each component  $U_i(t)$ ,  $i = 1, \dots, N$ , of the approximate m(c/d)G( $q$ ) solution  $U(t)$  of (1.1) is a piecewise polynomial on a partition of  $(0, T]$  into  $M_i$  subintervals. Subinterval  $j$  for component  $i$  is denoted by  $I_{ij} = (t_{i,j-1}, t_{ij}]$ , and the length of the subinterval is given by the *local time step*  $k_{ij} = t_{ij} - t_{i,j-1}$ . This is illustrated in Figure 1. On each subinterval  $I_{ij}$ ,  $U_i|_{I_{ij}}$  is a polynomial of degree  $q_{ij}$  and we refer to  $(I_{ij}, U_i|_{I_{ij}})$  as an *element*.

Furthermore, we shall assume that the interval  $(0, T]$  is partitioned into blocks between certain synchronized time levels  $0 = T_0 < T_1 < \dots < T_M = T$ . We refer to the set of intervals  $\mathcal{T}_n$  between two synchronized time levels  $T_{n-1}$  and  $T_n$  as a *time slab*:

$$\mathcal{T}_n = \{I_{ij} : T_{n-1} \leq t_{i,j-1} < t_{ij} \leq T_n\}.$$

We denote the length of a time slab by  $K_n = T_n - T_{n-1}$ . We also refer to the entire collection of intervals  $I_{ij}$  as the partition  $\mathcal{T}$ .

Since different components use different time steps, a local interval  $I_{ij}$  may contain nodal points for other components, that is, some  $t_{i'j'} \in (t_{i,j-1}, t_{ij})$ . We denote the set of such internal nodes on a local interval  $I_{ij}$  by  $\mathcal{N}_{ij}$ .

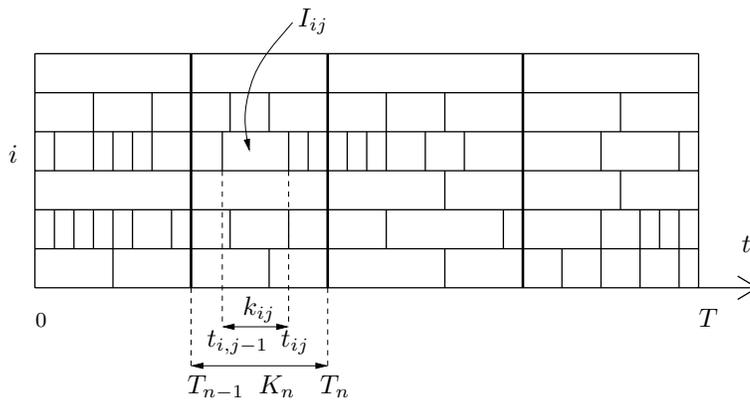


FIG. 1. Individual partitions of the interval  $(0, T]$  for different components. Elements between common synchronized time levels are organized in time slabs. In this example, we have  $N = 6$  and  $M = 4$ .

**1.3. Outline of the paper.** The outline of this paper is as follows. In section 2, we give the full definition of the multiadaptive Galerkin methods  $\text{mcG}(q)$  and  $\text{mdG}(q)$ . We also introduce the dual methods  $\text{mcG}(q)^*$  and  $\text{mdG}(q)^*$ , which are of importance to the a priori error analysis. In sections 3 and 4, respectively, we then prove existence and stability of the discrete solutions as defined in section 2.

In section 5, we prove the interpolation estimates that we later use to prove the a priori error estimates in section 6. Proving the interpolation estimates is technically challenging, since the function to be interpolated may be discontinuous within the interval of interpolation. To measure the regularity of the interpolated function, it is then necessary to take into consideration the size of the jump in function value and derivatives at each point of discontinuity.

Finally, in section 7, we present some numerical evidence for the a priori error estimates by solving a simple model problem and showing that we obtain the predicted convergence rates,  $k^{2q}$  and  $k^{2q+1}$ , respectively, for the  $\text{mcG}(q)$  and  $\text{mdG}(q)$  methods.

**2. Definition of methods.** In this section, we give the definitions of the multiadaptive Galerkin methods  $\text{mcG}(q)$  and  $\text{mdG}(q)$ . The multiadaptive methods are obtained as extensions of the standard (monoadaptive) Galerkin methods  $\text{cG}(q)$  and  $\text{dG}(q)$  by extending the trial and test spaces to allow individual time step sequences for different components.

As an important tool for the a priori error analysis in section 6, we also introduce the discrete dual problem and the discrete dual methods  $\text{mcG}(q)^*$  and  $\text{mdG}(q)^*$ .

**2.1. Multiadaptive continuous Galerkin,  $\text{mcG}(q)$ .** To formulate the  $\text{mcG}(q)$  method, we define the *trial space*  $V$  and the *test space*  $\hat{V}$  as

$$(2.1) \quad \begin{aligned} V &= \{v \in [\mathcal{C}([0, T])]^N : v_i|_{I_{ij}} \in \mathcal{P}^{q_{ij}}(I_{ij}), j = 1, \dots, M_i, i = 1, \dots, N\}, \\ \hat{V} &= \{v : v_i|_{I_{ij}} \in \mathcal{P}^{q_{ij}-1}(I_{ij}), j = 1, \dots, M_i, i = 1, \dots, N\}, \end{aligned}$$

where  $\mathcal{P}^q(I)$  denotes the linear space of polynomials of degree  $q$  on an interval  $I \subset \mathbb{R}$ . In other words,  $V$  is the space of vector-valued continuous piecewise polynomials of degree  $q = (q_i(t))$  with  $q_i(t) \geq 1$  on the partition  $\mathcal{T}$ , and  $\hat{V}$  is the space of vector-valued (possibly discontinuous) piecewise polynomials of degree  $q - 1 = (q_i(t) - 1)$  on the same partition.

We now define the mcG( $q$ ) method for (1.1) as follows: Find  $U \in V$  with  $U(0) = u_0$  such that

$$(2.2) \quad \int_0^T (\dot{U}, v) dt = \int_0^T (f(U, \cdot), v) dt \quad \forall v \in \hat{V},$$

where  $(\cdot, \cdot)$  denotes the  $\mathbb{R}^N$  inner product. With a suitable choice of test function  $v$ , it follows that the global problem (2.2) can be restated as a sequence of successive local problems for each component: For  $i = 1, \dots, N, j = 1, \dots, M_i$ , find  $U_i|_{I_{ij}} \in \mathcal{P}^{q_{ij}}(I_{ij})$  with  $U_i(t_{i,j-1})$  given such that

$$(2.3) \quad \int_{I_{ij}} \dot{U}_i v dt = \int_{I_{ij}} f_i(U, \cdot) v dt \quad \forall v \in \mathcal{P}^{q_{ij}-1}(I_{ij}),$$

where the initial condition is specified for  $i = 1, \dots, N$  by  $U_i(0) = u_i(0)$ .

We define the residual  $R$  of the approximate solution  $U$  by  $R_i(U, t) = \dot{U}_i(t) - f_i(U(t), t)$ . In terms of the residual, we can rewrite (2.3) in the form

$$(2.4) \quad \int_{I_{ij}} R_i(U, \cdot) v dt = 0 \quad \forall v \in \mathcal{P}^{q_{ij}-1}(I_{ij}), \quad j = 1, \dots, M_i, \quad i = 1, \dots, N,$$

that is, the residual is orthogonal to the test space on each local interval. We refer to (2.4) as the *Galerkin orthogonality* of the mcG( $q$ ) method.

**2.2. Multiadaptive discontinuous Galerkin, mdG( $q$ ).** For mdG( $q$ ), we define the trial and test spaces by

$$(2.5) \quad V = \hat{V} = \{v : v_i|_{I_{ij}} \in \mathcal{P}^{q_{ij}}(I_{ij}), j = 1, \dots, M_i, i = 1, \dots, N\},$$

that is, both trial and test functions are vector-valued (possibly discontinuous) piecewise polynomials of degree  $q = (q_i(t))$  with  $q_i(t) \geq 0$  on the partition  $\mathcal{T}$ . By definition, the mdG( $q$ ) solution  $U \in V$  is left-continuous.

We now define the mdG( $q$ ) method for (1.1) as follows: Find  $U \in V$  with  $U(0^-) = u_0$  such that

$$(2.6) \quad \sum_{i=1}^N \sum_{j=1}^{M_i} \left[ [U_i]_{i,j-1} v_i(t_{i,j-1}^+) + \int_{I_{ij}} \dot{U}_i v_i dt \right] = \int_0^T (f(U, \cdot), v) dt \quad \forall v \in \hat{V},$$

where  $[U_i]_{i,j-1} = U_i(t_{i,j-1}^+) - U_i(t_{i,j-1}^-)$  denotes the jump in  $U_i(t)$  across the node  $t = t_{i,j-1}$ , and where  $v(t^+) = \lim_{s \rightarrow t^+} v(s)$ .

The mdG( $q$ ) method in local form, corresponding to (2.3), reads as follows: For  $i = 1, \dots, N, j = 1, \dots, M_i$ , find  $U_i|_{I_{ij}} \in \mathcal{P}^{q_{ij}}(I_{ij})$  such that

$$(2.7) \quad [U_i]_{i,j-1} v(t_{i,j-1}) + \int_{I_{ij}} \dot{U}_i v dt = \int_{I_{ij}} f_i(U, \cdot) v dt \quad \forall v \in \mathcal{P}^{q_{ij}}(I_{ij}),$$

where the initial condition is specified for  $i = 1, \dots, N$  by  $U_i(0^-) = u_i(0)$ .

The residual  $R$  is defined on the inner of each local interval  $I_{ij}$  by  $R_i(U, t) = \dot{U}_i(t) - f_i(U(t), t)$ . In terms of the residual, (2.7) can be restated in the form

$$(2.8) \quad [U_i]_{i,j-1} v(t_{i,j-1}^+) + \int_{I_{ij}} R_i(U, \cdot) v dt = 0 \quad \forall v \in \mathcal{P}^{q_{ij}}(I_{ij})$$

for  $j = 1, \dots, M_i, i = 1, \dots, N$ . We refer to (2.8) as the Galerkin orthogonality of the mdG( $q$ ) method.

**2.3. The dual problem.** The dual problem is the standard tool for error analysis, a priori or a posteriori, of Galerkin finite element methods for the numerical solution of differential equations; see [8, 3]. For the a posteriori error analysis of the multiadaptive Galerkin methods mcG( $q$ ) and mdG( $q$ ) in [32], we formulate a continuous dual problem. For the a priori error analysis of this paper, we formulate instead a discrete dual problem. The discrete dual problem was first introduced for the family of discontinuous Galerkin methods dG( $q$ ) in [16]. As we shall see, the discrete dual problem can be expressed as a Galerkin method for a continuous problem.

The discrete dual solution  $\Phi : [0, T] \rightarrow \mathbb{R}^N$  is a Galerkin approximation of the exact solution  $\phi : [0, T] \rightarrow \mathbb{R}^N$  of the continuous dual problem

$$(2.9) \quad \begin{aligned} -\dot{\phi}(t) &= J^\top(\pi u, U, t)\phi(t) + g(t), \quad t \in [0, T], \\ \phi(T) &= \psi, \end{aligned}$$

where  $\pi u$  is an interpolant or a projection of the exact solution  $u$  of (1.1),  $g : [0, T] \rightarrow \mathbb{R}^N$  is a given function,  $\psi \in \mathbb{R}^N$  is a given initial condition, and

$$(2.10) \quad J^\top(\pi u, U, t) = \left( \int_0^1 \frac{\partial f}{\partial u}(s\pi u(t) + (1-s)U(t), t) ds \right)^\top,$$

that is, an appropriate mean value of the transpose of the Jacobian of the right-hand side  $f(\cdot, t)$  evaluated at  $\pi u(t)$  and  $U(t)$ . Note that by the chain rule, we have

$$(2.11) \quad J(\pi u, U, \cdot)(U - \pi u) = f(U, \cdot) - f(\pi u, \cdot).$$

The data  $(\psi, g)$  of the dual problem allow us to obtain error estimates for different functionals  $L_{\psi, g}$  of the error  $e = U - u$ .

We define below two new Galerkin methods for the dual problem (2.9): the dual methods mcG( $q$ )<sup>\*</sup> and mdG( $q$ )<sup>\*</sup>. We will later use the mcG( $q$ )<sup>\*</sup> method to express the error of the mcG( $q$ ) solution of (1.1) in terms of the mcG( $q$ )<sup>\*</sup> solution of (2.9). Similarly, we will express the error of the mdG( $q$ ) solution of (1.1) in terms of the mdG( $q$ )<sup>\*</sup> solution of (2.9).

**2.4. Multiadaptive dual continuous Galerkin, mcG( $q$ )<sup>\*</sup>.** In the formulation of the dual method of mcG( $q$ ), we interchange the trial and test spaces of mcG( $q$ ). With the same definitions of  $V$  and  $\hat{V}$  as in (2.1), we thus define the mcG( $q$ )<sup>\*</sup> method for (2.9) as follows: Find  $\Phi \in \hat{V}$  with  $\Phi(T^+) = \psi$  such that

$$(2.12) \quad \int_0^T (\dot{v}, \Phi) dt = \int_0^T (J(\pi u, U, \cdot)v, \Phi) + L_{\psi, g}(v)$$

for all  $v \in V$  with  $v(0) = 0$ , where

$$(2.13) \quad L_{\psi, g}(v) \equiv (v(T), \psi) + \int_0^T (v, g) dt.$$

Notice the extra condition that the test functions should vanish at  $t = 0$ , which is introduced to make the dimension of the test space equal to the dimension of the trial space. Integrating by parts, (2.12) can alternatively be expressed in the form

$$(2.14) \quad \sum_{i=1}^N \sum_{j=1}^{M_i} \left[ -[\Phi_i]_{ij} v_i(t_{ij}) - \int_{I_{ij}} \dot{\Phi}_i v_i dt \right] = \int_0^T (J^\top(\pi u, U, \cdot)\Phi + g, v) dt.$$

**2.5. Multiadaptive dual discontinuous Galerkin, mdG(q)\*.** With the same definitions of  $V$  and  $\hat{V}$  as in (2.5), we define the mdG(q)\* method for (2.9) as follows: Find  $\Phi \in \hat{V}$  with  $\Phi(T^+) = \psi$  such that

$$(2.15) \quad \sum_{i=1}^N \sum_{j=1}^{M_i} \left[ [v_i]_{i,j-1} \Phi_i(t_{i,j-1}^+) + \int_{I_{ij}} \dot{v}_i \Phi_i dt \right] = \int_0^T (J(\pi u, U, \cdot)v, \Phi) dt + L_{\psi,g}(v)$$

for all  $v \in V$  with  $v(0^-) = 0$ . Integrating by parts, (2.15) can alternatively be expressed in the form

$$(2.16) \quad \sum_{i=1}^N \sum_{j=1}^{M_i} \left[ -[\Phi_i]_{ij} v_i(t_{ij}^-) - \int_{I_{ij}} \dot{\Phi}_i v_i dt \right] = \int_0^T (J^\top(\pi u, U, \cdot)\Phi + g, v) dt.$$

**3. Existence of solutions.** To prove existence of the discrete mcG(q), mdG(q), mcG(q)\*, and mdG(q)\* solutions defined in the previous section, we formulate fixed point iterations for the construction of solutions. Existence then follows from the Banach fixed point theorem if the time steps are sufficiently small.

LEMMA 3.1 (fixed point iteration). *Let  $\mathcal{T}_n$  be a time slab with synchronized time levels  $T_{n-1}$  and  $T_n$ . With time reversed for the dual methods (to simplify the notation), the mcG(q), mdG(q), mcG(q)\*, and mdG(q)\* methods can all be expressed in the following form: For all  $I_{ij} \in \mathcal{T}_n$ , find  $\{\xi_{ijn}\}$  (the degrees of freedom for  $U_i$  on  $I_{ij}$ ) such that*

$$(3.1) \quad \xi_{ijn} = u_i(0) + \int_0^{t_{i,j-1}} f_i(U, \cdot) dt + \int_{I_{ij}} w_n^{[q_{ij}]}(\tau_{ij}(t)) f_i(U, \cdot) dt,$$

where  $\tau_{ij}(t) = (t - t_{i,j-1}) / (t_{ij} - t_{i,j-1})$  and  $\{w_n^{[q_{ij}]}\}$  is a set of polynomial weight functions on  $[0, 1]$ .

*Proof.* The result follows from the definitions of the mcG(q), mdG(q), mcG(q)\*, and mdG(q)\* methods, using an appropriate basis for the trial and test spaces. See [34] for details.  $\square$

THEOREM 3.2 (existence of solutions). *Let  $K = \max K_n$  be the maximum time slab length and define the Lipschitz constant  $L_f > 0$  by*

$$(3.2) \quad \|f(x, t) - f(y, t)\|_{l_\infty} \leq L_f \|x - y\|_{l_\infty} \quad \forall t \in [0, T] \quad \forall x, y \in \mathbb{R}^N.$$

If now

$$(3.3) \quad KCL_f < 1,$$

where  $C = C(q) > 0$  is a constant depending only on the order and method, the fixed point iteration (3.1) converges to the unique solution of (2.2), (2.6), (2.12), and (2.15), respectively.

*Proof.* The result follows by Lemma 3.1 and an application of the Banach fixed point theorem. See [34] for details.  $\square$

**4. Stability of solutions.** Write the dual problem (2.9) for  $\phi = \phi(t)$  in the form

$$(4.1) \quad \begin{aligned} -\dot{\phi}(t) + A^\top(t)\phi(t) &= g, & t \in [0, T], \\ \phi(T) &= \psi. \end{aligned}$$

For simplicity, we consider only the case  $g = 0$ . With  $w(t) = \phi(T - t)$ , we have  $\dot{w}(t) = -\dot{\phi}(T - t) = -A^\top(T - t)w(t)$ , and so (4.1) can be written as a forward problem for  $w$  in the form

$$(4.2) \quad \begin{aligned} \dot{w}(t) + B(t)w(t) &= 0, \quad t \in (0, T], \\ w(0) &= w_0, \end{aligned}$$

where  $w_0 = \psi$  and  $B(t) = A^\top(T - t)$ . Below,  $w$  represents either  $u$  or  $\phi(T - \cdot)$  and, correspondingly,  $W$  represents either the discrete mc/dG( $q$ ) approximation  $U$  of  $u$  or the discrete mc/dG( $q$ )<sup>\*</sup> approximation  $\Phi$  of  $\phi$ .

**4.1. A general exponential estimate.** The general exponential stability estimate is based on the following version of the discrete Gronwall inequality.

LEMMA 4.1 (discrete Gronwall inequality). *Assume that  $z, a : \mathbb{N} \rightarrow \mathbb{R}$  are non-negative,  $a(m) \leq 1/2$  for all  $m$ , and  $z(n) \leq C + \sum_{m=1}^n a(m)z(m)$  for all  $n$ . Then  $z(n) \leq 2C \exp(\sum_{m=1}^{n-1} 2a(m))$  for  $n = 1, 2, \dots$ .*

*Proof.* By a standard discrete Gronwall inequality [38],  $z(n) \leq C \exp(\sum_{m=0}^{n-1} a(m))$  if  $z(n) \leq C + \sum_{m=0}^{n-1} a(m)z(m)$  for  $n \geq 1$  and  $z(0) \leq C$ . Here,  $(1 - a(n))z(n) \leq C + \sum_{m=1}^{n-1} a(m)z(m)$ , and so  $z(n) \leq 2C + \sum_{m=1}^{n-1} 2a(m)z(m)$ , since  $1 - a(n) \geq 1/2$ . The result now follows if we take  $a(0) = z(0) = 0$ .  $\square$

THEOREM 4.2 (stability estimate). *Let  $W$  be the mcG( $q$ ), mdG( $q$ ), mcG( $q$ )<sup>\*</sup>, or mdG( $q$ )<sup>\*</sup> solution of (4.2). Then there is a constant  $C = C(q)$ , depending only on the highest order  $\max q_{ij}$ , such that if  $K_n C \|B\|_{L_\infty([T_{n-1}, T_n], l_p)} \leq 1$  for  $n = 1, \dots, M$ , then*

$$(4.3) \quad \|W\|_{L_\infty([T_{n-1}, T_n], l_p)} \leq C \|w_0\|_{l_p} \exp\left(\sum_{m=1}^{n-1} K_m C \|B\|_{L_\infty([T_{m-1}, T_m], l_p)}\right)$$

for  $n = 1, \dots, M$ ,  $1 \leq p \leq \infty$ .

*Proof.* By Lemma 3.1, we can write the mcG( $q$ ), mdG( $q$ ), mcG( $q$ )<sup>\*</sup>, and mdG( $q$ )<sup>\*</sup> methods in the form  $\xi_{ijn'} = w_i(0) + \int_0^{t_{i,j-1}} f_i(W, \cdot) dt + \int_{I_{ij}} w_{n'}^{[q_{ij}]}(\tau_{ij}(t)) f_i(W, \cdot) dt$ . Applied to the linear model problem (4.2), we have  $\xi_{ijn'} = w_i(0) - \int_0^{t_{i,j-1}} (BW)_i dt - \int_{I_{ij}} w_{n'}^{[q_{ij}]}(\tau_{ij}(t))(BW)_i dt$ , and so

$$\begin{aligned} |\xi_{ijn'}| &\leq |w_i(0)| + \left| \int_0^{t_{i,j-1}} (BW)_i dt \right| + \left| \int_{I_{ij}} w_{n'}^{[q_{ij}]}(\tau_{ij}(t))(BW)_i dt \right| \\ &\leq |w_i(0)| + C \int_0^{t_{ij}} |(BW)_i| dt \leq |w_i(0)| + C \int_0^{T_n} |(BW)_i| dt, \end{aligned}$$

where  $T_n$  is smallest synchronized time level for which  $t_{ij} \leq T_n$ . It now follows that for all  $t \in [T_{n-1}, T_n]$ , we have  $|W_i(t)| \leq C|w_i(0)| + C \int_0^{T_n} |(BW)_i| dt$ , and so

$$\|W(t)\|_{l_p} \leq C \|w_0\|_{l_p} + C \int_0^{T_n} \|BW\|_{l_p} dt = C \|w_0\|_{l_p} + C \sum_{m=1}^n \int_{T_{m-1}}^{T_m} \|BW\|_{l_p} dt.$$

The result now follows by letting  $\bar{W}_n = \|W\|_{L_\infty([T_{n-1}, T_n], l_p)}$ .  $\square$

REMARK 4.1. See [34] for an extension to multiadaptive time-stepping of the strong stability estimate Lemma 6.1 for parabolic problems in [11].

**5. Interpolation estimates.** In this section, we introduce a pair of carefully chosen interpolants,  $\pi_{cG}^{[q]}$  and  $\pi_{dG}^{[q]}$ , which are central to the a priori error analysis of the mcG( $q$ ) and mdG( $q$ ) methods. The interpolants are defined in section 5.1. In section 5.2, we discuss the interpolation of piecewise smooth functions, that is, the interpolation of functions which may be discontinuous within the interval of interpolation, and then present the basic general interpolation estimates for the two interpolants  $\pi_{cG}^{[q]}$  and  $\pi_{dG}^{[q]}$ .

For the a priori error analysis of the mcG( $q$ ) and mdG( $q$ ) methods, we will also need a special interpolation estimate for the function  $\varphi = J^T \Phi$ , where  $J$  is the Jacobian of the right-hand side  $f$  of (1.1) and  $\Phi$  is the discrete dual solution as defined in section 2, including estimates for the size of the jump in function value and derivatives for the function  $\varphi$  at points of discontinuity. These estimates are proved in section 5.3, based on a representation formula for the mcG( $q$ ) and mdG( $q$ ) solutions of (1.1).

**5.1. Interpolants.** The interpolant  $\pi_{cG}^{[q]} : V \rightarrow \mathcal{P}^q([a, b])$  is defined by the following conditions:

$$(5.1) \quad \begin{aligned} \pi_{cG}^{[q]}v(a) &= v(a) \quad \text{and} \quad \pi_{cG}^{[q]}v(b) = v(b), \\ \int_a^b (v - \pi_{cG}^{[q]}v)w \, dx &= 0 \quad \forall w \in \mathcal{P}^{q-2}([a, b]), \end{aligned}$$

where  $V$  denotes the set of functions that are piecewise  $\mathcal{C}^{q+1}$  and bounded on  $[a, b]$ . In other words,  $\pi_{cG}^{[q]}v$  is the polynomial of degree  $q$  that interpolates  $v$  at the end-points of the interval  $[a, b]$  and additionally satisfies  $q - 1$  projection conditions. This is illustrated in Figure 2. We also define the dual interpolant  $\pi_{cG^*}^{[q]}$  as the standard  $L_2$ -projection onto  $\mathcal{P}^{q-1}([a, b])$ .

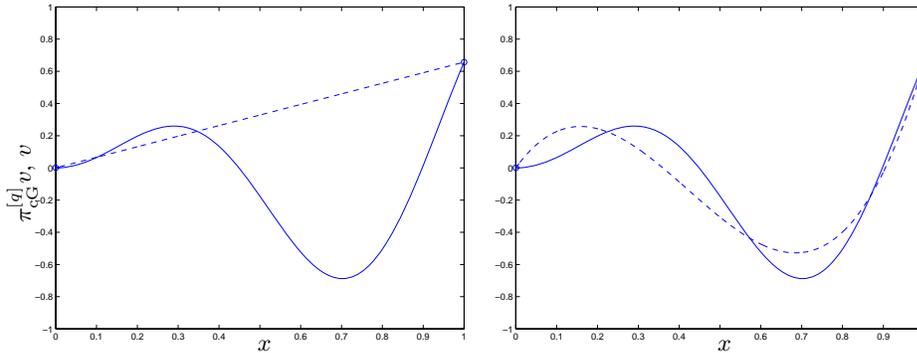


FIG. 2. The interpolant  $\pi_{cG}^{[q]}v$  (dashed) of the function  $v(x) = x \sin(7x)$  (solid) on  $[0, 1]$  for  $q = 1$  (left) and  $q = 3$  (right).

The interpolant  $\pi_{dG}^{[q]} : V \rightarrow \mathcal{P}^q([a, b])$  is defined by the following conditions:

$$(5.2) \quad \begin{aligned} \pi_{dG}^{[q]}v(b) &= v(b), \\ \int_a^b (v - \pi_{dG}^{[q]}v)w \, dx &= 0 \quad \forall w \in \mathcal{P}^{q-1}([a, b]), \end{aligned}$$

that is,  $\pi_{dG}^{[q]}v$  is the polynomial of degree  $q$  that interpolates  $v$  at the right end-point of the interval  $[a, b]$  and additionally satisfies  $q$  projection conditions. This is illustrated

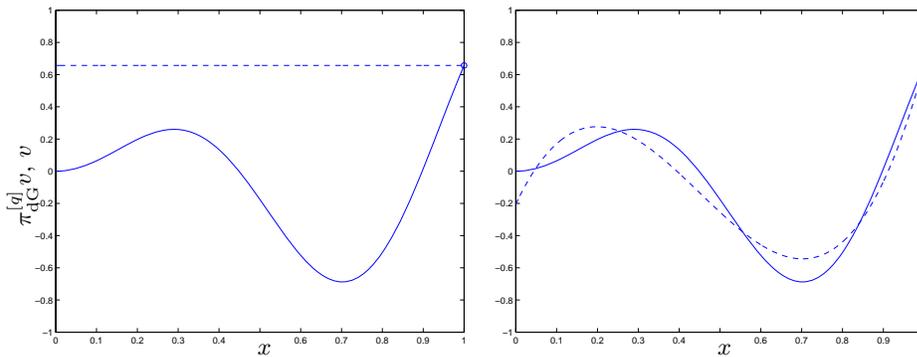


FIG. 3. The interpolant  $\pi_{dG}^{[q]}v$  (dashed) of the function  $v(x) = x \sin(7x)$  (solid) on  $[0, 1]$  for  $q = 0$  (left) and  $q = 3$  (right).

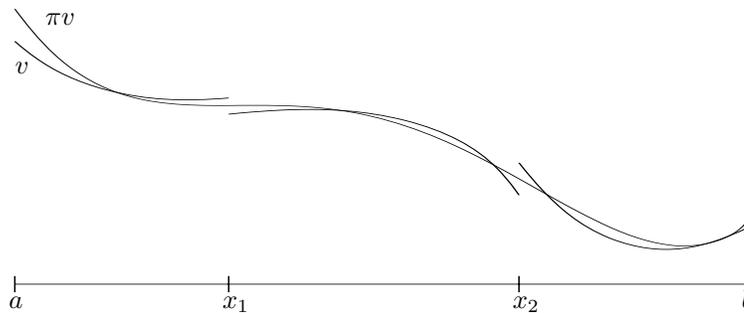


FIG. 4. A piecewise smooth function  $v$  and its interpolant  $\pi v$ .

in Figure 3. The dual interpolant  $\pi_{dG^*}^{[q]}$  is defined similarly, with the difference being that the left end-point  $x = a$  is used for interpolation.

**5.2. Basic interpolation estimates.** To estimate the size of the interpolation error  $\pi v - v$  for a given function  $v$ , we express the interpolation error in terms of the regularity of  $v$  and the length of the interpolation interval,  $k = b - a$ . Specifically, when  $v \in \mathcal{C}^{q+1}([a, b]) \subset V$  for some  $q \geq 0$ , we obtain estimates of the form

$$(5.3) \quad \|(\pi v)^{(p)} - v^{(p)}\| \leq C k^{q+1-p} \|v^{(q+1)}\|, \quad p = 0, \dots, q + 1,$$

where  $\|\cdot\| = \|\cdot\|_{L^\infty([a,b])}$  denotes the maximum norm on  $[a, b]$ . This estimate is a simple consequence of the Peano kernel theorem [40] if one can show that the interpolant  $\pi : V \rightarrow \mathcal{P}^q([a, b]) \subset V$  is linear and bounded on  $V$  and that  $\pi$  is exact on  $\mathcal{P}^q([a, b]) \subset V$ , that is,  $\pi v = v$  for all  $v \in \mathcal{P}^q([a, b])$ .

In the general case, where the interpolated function  $v$  is only piecewise smooth (see Figure 4), we also need to include the size of the jump  $[v^{(p)}]_x$  in function value and derivatives at each point  $x$  of discontinuity within  $(a, b)$  to measure the regularity of the interpolated function  $v$ . In [34], we prove the following extensions of the basic estimate (5.3).

**LEMMA 5.1.** *If  $\pi$  is linear and bounded on  $V$  and is exact on  $\mathcal{P}^q([a, b]) \subset V$ , then there is a constant  $C = C(q) > 0$  such that for all  $v$  piecewise  $\mathcal{C}^{q+1}$  on  $[a, b]$  with*

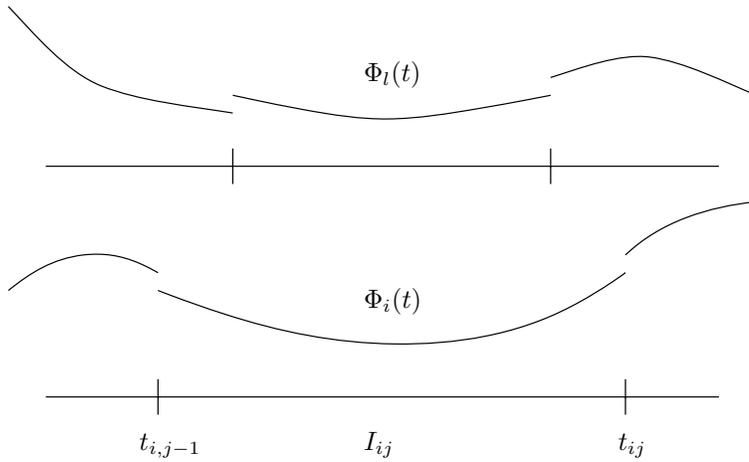


FIG. 5. If some other component  $l \neq i$  has a node within  $I_{ij}$ , then  $\Phi_l$  may be discontinuous within  $I_{ij}$ , causing  $\varphi_i$  to be discontinuous within  $I_{ij}$ .

discontinuities at  $a < x_1 < \dots < x_n < b$ ,

$$(5.4) \quad \|(\pi v)^{(p)} - v^{(p)}\| \leq Ck^{r+1-p} \|v^{(r+1)}\| + C \sum_{j=1}^n \sum_{m=0}^r k^{m-p} |[v^{(m)}]_{x_j}|$$

for  $p = 0, \dots, r + 1$ ,  $r = 0, \dots, q$ .

LEMMA 5.2. If  $\pi$  is linear and bounded on  $V$  and is exact on  $\mathcal{P}^q([a, b]) \subset V$ , then there is a constant  $C = C(q) > 0$  such that for all  $v$  piecewise  $\mathcal{C}^{q+1}$  on  $[a, b]$  with discontinuities at  $a < x_1 < \dots < x_n < b$ ,

$$(5.5) \quad \|(\pi v)^{(p)}\| \leq C \|v^{(p)}\| + C \sum_{j=1}^n \sum_{m=0}^{p-1} k^{m-p} |[v^{(m)}]_{x_j}|$$

for  $p = 0, \dots, q$ .

Lemmas 5.1 and 5.2 apply to both the  $\pi_{\text{cG}}^{[q]}$  interpolant (for  $q \geq 1$ ) and the  $\pi_{\text{dG}}^{[q]}$  interpolant (for  $q \geq 0$ ) defined in section 5.1. The linearity of both interpolants follows directly from the definition of the interpolants. The proofs that both interpolants are bounded and exact on  $\mathcal{P}^q([a, b])$  are given in detail in [34] and [35].

**5.3. A special interpolation estimate.** To prove a priori error estimates for mcG( $q$ ) and mdG( $q$ ) in section 6, we need to estimate the interpolation error  $\pi\varphi - \varphi$  for the function  $\varphi$  defined by

$$(5.6) \quad \varphi_i = (J^\top(\pi u, u, \cdot)\Phi)_i = \sum_{l=1}^N J_{li}(\pi u, u, \cdot)\Phi_l, \quad i = 1, \dots, N.$$

We note that  $\varphi_i$  may be discontinuous within  $I_{ij}$  if  $I_{ij}$  contains a node for some other component, which is generally the case. This is illustrated in Figure 5. Note that on the right-hand side  $f$  is linearized around a mean value of  $\pi u$  and  $u$ .

An interpolation estimate for  $\pi\varphi - \varphi$  follows directly from Lemma 5.1. To use this estimate, we need to estimate the size of the jump in function value and derivatives at

each internal node  $t_{ij}$  of the partition  $\mathcal{T}$ . To obtain this estimate, we need to make a number of additional assumptions on the right-hand side  $f$  of (1.1) and the partition  $\mathcal{T}$ . These assumptions are discussed in section 5.3.2. Based on the assumptions and the representation formula presented in section 5.3.1, we obtain the jump estimates in section 5.3.3 and, finally, in section 5.3.4, the interpolation estimate for  $\varphi$ .

**5.3.1. A representation formula.** The proof of jump estimates for the multi-adaptive Galerkin methods mcG( $q$ ) and mdG( $q$ ) is based on expressing the solutions as certain interpolants. These representations are obtained as follows. Let  $U$  be the mcG( $q$ ) or mdG( $q$ ) solution of (1.1) and define, for  $i = 1, \dots, N$ ,

$$(5.7) \quad \tilde{U}_i(t) = u_i(0) + \int_0^t f_i(U(s), s) ds.$$

Similarly, for  $\Phi$  the mcG( $q$ )<sup>\*</sup> or mdG( $q$ )<sup>\*</sup> solution of (2.9), we define, for  $i = 1, \dots, N$ ,

$$(5.8) \quad \tilde{\Phi}_i(t) = \psi_i + \int_t^T f_i^*(\Phi(s), s) ds,$$

where  $f^*(\Phi, \cdot) = J^\top(\pi u, U, \cdot)\Phi + g$ . We note that  $\dot{\tilde{U}} = f(U, \cdot)$  and  $-\dot{\tilde{\Phi}} = f^*(\Phi, \cdot)$ .

It now turns out that  $U$  can be expressed as an interpolant of  $\tilde{U}$ . Similarly,  $\Phi$  can be expressed as an interpolant of  $\tilde{\Phi}$ . We present these representations in Lemmas 5.3 and 5.4. We remind the reader about the interpolants  $\pi_{cG}^{[q]}$ ,  $\pi_{cG^*}^{[q]}$ ,  $\pi_{dG}^{[q]}$ , and  $\pi_{dG^*}^{[q]}$ , defined in section 5.1.

LEMMA 5.3. *The mcG( $q$ ) solution  $U$  of (1.1) can be expressed in the form  $U = \pi_{cG}^{[q]}\tilde{U}$ . Similarly, the mcG( $q$ )<sup>\*</sup> solution  $\Phi$  of (2.9) can be expressed in the form  $\Phi = \pi_{cG^*}^{[q]}\tilde{\Phi}$ , that is,  $U_i = \pi_{cG}^{[q]}\tilde{U}_i$  and  $\Phi_i = \pi_{cG^*}^{[q]}\tilde{\Phi}_i$  on each local interval  $I_{ij}$ .*

*Proof.* The representation formulas follow by the definitions of the mcG( $q$ ) and mcG( $q$ )<sup>\*</sup> methods and the interpolants  $\pi_{cG}^{[q]}$  and  $\pi_{cG^*}^{[q]}$ . See [34] for details.  $\square$

LEMMA 5.4. *The mdG( $q$ ) solution  $U$  of (1.1) can be expressed in the form  $U = \pi_{dG}^{[q]}\tilde{U}$ . Similarly, the mdG( $q$ )<sup>\*</sup> solution  $\Phi$  of (2.9) can be expressed in the form  $\Phi = \pi_{dG^*}^{[q]}\tilde{\Phi}$ , that is,  $U_i = \pi_{dG}^{[q]}\tilde{U}_i$  and  $\Phi_i = \pi_{dG^*}^{[q]}\tilde{\Phi}_i$  on each local interval  $I_{ij}$ .*

*Proof.* The representation formulas follow by the definitions of the mdG( $q$ ) and mdG( $q$ )<sup>\*</sup> methods and the interpolants  $\pi_{dG}^{[q]}$  and  $\pi_{dG^*}^{[q]}$ . See [34] for details.  $\square$

**5.3.2. Assumptions.** To estimate the size of the jump in function value and derivatives for the function  $\varphi$  defined in (5.6), we make the following assumptions. Given a time slab  $\mathcal{T}$ , assume that for each pair of local intervals  $I_{ij}$  and  $I_{mn}$  within the time slab, we have

$$(A1) \quad q_{ij} = q_{mn} = \bar{q}$$

and

$$(A2) \quad k_{ij} > \alpha k_{mn}$$

for some  $\bar{q} \geq 0$  and some  $\alpha \in (0, 1)$ . The dependence on  $\alpha$  in the error estimates is weak (see Remark 5.1), so assumption (A2) does not prevent multiadaptivity.

We also assume that the problem (1.1) is autonomous,

$$(A3) \quad \partial f_i / \partial t = 0, \quad i = 1, \dots, N,$$

noting that the dual problem nevertheless will be nonautonomous in general. Furthermore, we assume that

$$(A4) \quad \|f_i\|_{D^{\bar{q}+1}(\mathcal{T})} < \infty, \quad i = 1, \dots, N,$$

where  $\|\cdot\|_{D^p(\mathcal{T})}$  is defined for  $v : \mathbb{R}^N \rightarrow \mathbb{R}$  and  $p \geq 0$  by  $\|v\|_{D^p(\mathcal{T})} = \max_{n=0, \dots, p} \|D^n v\|_{L_\infty(\mathcal{T}, l_\infty)}$ , with the norm  $\|D^n v\|_{L_\infty(\mathcal{T}, l_\infty)}$  defined by  $\|D^n v w^1 \cdots w^n\|_{L_\infty(\mathcal{T})} \leq \|D^n v\|_{L_\infty(\mathcal{T}, l_\infty)} \|w^1\|_{l_\infty} \cdots \|w^n\|_{l_\infty}$  for all  $w^1, \dots, w^n \in \mathbb{R}^N$ , and  $D^n v$  the  $n$ th-order tensor given by

$$D^n v w^1 \cdots w^n = \sum_{i_1=1}^N \cdots \sum_{i_n=1}^N \frac{\partial^n v}{\partial x_{i_1} \cdots \partial x_{i_n}} w_{i_1}^1 \cdots w_{i_n}^n.$$

Furthermore, we choose  $C_f \geq \max_{i=1, \dots, N} \|f_i\|_{D^{\bar{q}+1}(\mathcal{T})}$  such that

$$(5.9) \quad \|d^p/dt^p(\partial f/\partial u)^\top(x(t))\|_{l_\infty} \leq C_f C_x^p$$

for  $p = 0, \dots, \bar{q}$ , and

$$(5.10) \quad \|[d^p/dt^p(\partial f/\partial u)^\top(x(t))]_t\|_{l_\infty} \leq C_f \sum_{n=0}^p C_x^{p-n} \|[x^{(n)}]_t\|_{l_\infty}$$

for  $p = 0, \dots, \bar{q} - 1$  and any given  $x : \mathbb{R} \rightarrow \mathbb{R}^N$ , where  $C_x > 0$  denotes a constant such that  $\|x^{(n)}\|_{L_\infty(\mathcal{T}, l_\infty)} \leq C_x^n$  for  $n = 1, \dots, p$ . Note that  $C_f = C_f(t)$  defines a piecewise constant function on the partition  $0 = T_0 < T_1 < \cdots < T_M = T$ . Note also that assumption (A4) implies that each  $f_i$  is bounded by  $C_f$ .

We further assume that there is a constant  $c_k > 0$  such that

$$(A5) \quad k_{ij} C_f \leq c_k$$

for each local interval  $I_{ij}$ . We summarize the list of assumptions as follows:

- (A1) the local orders  $q_{ij}$  are equal within each time slab;
- (A2) the local time steps  $k_{ij}$  are semiuniform within each time slab;
- (A3)  $f$  is autonomous;
- (A4)  $f$  and its derivatives are bounded;
- (A5) the local time steps  $k_{ij}$  are small.

**5.3.3. Estimates of derivatives and jumps.** To estimate higher-order derivatives, we face the problem of taking higher-order derivatives of  $f(U(t), t)$  with respect to  $t$ . In Lemmas 5.5 and 5.6, we present basic estimates for composite functions  $v \circ x$  with  $v : \mathbb{R}^N \rightarrow \mathbb{R}$  and  $x : \mathbb{R} \rightarrow \mathbb{R}^N$ . The proofs are based on a straightforward application of the chain rule and Leibniz rule and are given in full detail in [34].

LEMMA 5.5. *Let  $v : \mathbb{R}^N \rightarrow \mathbb{R}$  be  $p \geq 0$  times differentiable in all its variables, let  $x : \mathbb{R} \rightarrow \mathbb{R}^N$  be  $p$  times differentiable, and let  $C_x > 0$  be a constant such that  $\|x^{(n)}\|_{L_\infty(\mathbb{R}, l_\infty)} \leq C_x^n$  for  $n = 1, \dots, p$ . Then there is a constant  $C = C(p) > 0$  such that*

$$(5.11) \quad \left\| \frac{d^p(v \circ x)}{dt^p} \right\|_{L_\infty(\mathbb{R})} \leq C \|v\|_{D^p(\mathbb{R})} C_x^p.$$

LEMMA 5.6. *Let  $v : \mathbb{R}^N \rightarrow \mathbb{R}$  be  $p + 1 \geq 1$  times differentiable in all its variables, let  $x : \mathbb{R} \rightarrow \mathbb{R}^N$  be  $p$  times differentiable, except possibly at some  $t \in \mathbb{R}$ , and let*

$C_x > 0$  be a constant such that  $\|x^{(n)}\|_{L_\infty(\mathbb{R}, l_\infty)} \leq C_x^n$  for  $n = 1, \dots, p$ . Then there is a constant  $C = C(p) > 0$  such that

$$(5.12) \quad \left\| \left[ \frac{d^p(v \circ x)}{dt^p} \right]_t \right\| \leq C \|v\|_{D^{p+1}(\mathbb{R})} \sum_{n=0}^p C_x^{p-n} \|[x^{(n)}]_t\|_{l_\infty}.$$

We now prove estimates for derivatives and jumps of the mcG( $q$ ) or mdG( $q$ ) solution  $U$  of the general nonlinear problem (1.1), under the assumptions listed in section 5.3.2. Similarly, one can obtain estimates for the discrete dual solution  $\Phi$  and the function  $\varphi$  defined in (5.6), from which the desired interpolation estimates follow.

To obtain estimates for the multiadaptive solution  $U$ , we first prove estimates for the function  $\tilde{U}$  defined in section 5.3.1. The estimates for  $U$  then follow by induction.

To simplify the estimates, we introduce the following notation. For given  $p > 0$ , let  $C_{U,p} \geq C_f$  be a constant such that

$$(5.13) \quad \|U^{(n)}\|_{L_\infty(\mathcal{T}, l_\infty)} \leq C_{U,p}^n, \quad n = 1, \dots, p.$$

For  $p = 0$ , we define  $C_{U,0} = C_f$ . Temporarily, we assume that there is a constant  $c'_k > 0$  such that for each  $p$ ,

$$(A5') \quad k_{ij} C_{U,p} \leq c'_k.$$

This assumption will be removed in Lemma 5.9. In the following lemma, we use assumptions (A1), (A3), and (A4) to derive estimates for  $\tilde{U}$  in terms of  $C_{U,p}$  and  $C_f$ .

LEMMA 5.7 (derivative and jump estimates for  $\tilde{U}$ ). *Let  $U$  be the mcG( $q$ ) or mdG( $q$ ) solution of (1.1) and define  $\tilde{U}$  as in (5.7). If assumptions (A1), (A3), and (A4) hold, then there is a constant  $C = C(\bar{q}) > 0$  such that*

$$(5.14) \quad \|\tilde{U}^{(p)}\|_{L_\infty(\mathcal{T}, l_\infty)} \leq C C_{U,p-1}^p, \quad p = 1, \dots, \bar{q} + 1,$$

and

$$(5.15) \quad \|\tilde{U}^{(p)}\|_{t_{i,j-1}} \Big|_{l_\infty} \leq C \sum_{n=0}^{p-1} C_{U,p-1}^{p-n} \|[U^{(n)}]_{t_{i,j-1}}\|_{l_\infty}, \quad p = 1, \dots, \bar{q} + 1,$$

for each local interval  $I_{ij}$ , where  $t_{i,j-1}$  is an internal node of the time slab  $\mathcal{T}$ .

*Proof.* By definition,  $\tilde{U}_i^{(p)} = \frac{d^{p-1}}{dt^{p-1}} f_i(U)$ , and so the results follow directly by Lemmas 5.5 and 5.6, noting that  $C_f \leq C_{U,p-1}$ .  $\square$

By Lemma 5.7, we now obtain the following estimate for the size of the jump in function value and derivatives for  $U$ .

LEMMA 5.8 (jump estimates for  $U$ ). *Let  $U$  be the mcG( $q$ ) or mdG( $q$ ) solution of (1.1). If assumptions (A1)–(A5) and (A5') hold, then there is a constant  $C = C(\bar{q}, c_k, c'_k, \alpha) > 0$  such that*

$$(5.16) \quad \|[U^{(p)}]_{t_{i,j-1}}\|_{l_\infty} \leq C k_{ij}^{r+1-p} C_{U,r}^{r+1}, \quad p = 0, \dots, r + 1, \quad r = 0, \dots, \bar{q},$$

for each local interval  $I_{ij}$ , where  $t_{i,j-1}$  is an internal node of the time slab  $\mathcal{T}$ .

*Proof.* The proof is by induction. We first note that at  $t = t_{i,j-1}$ , we have

$$\begin{aligned} [U_i^{(p)}]_t &= U_i^{(p)}(t^+) - \tilde{U}_i^{(p)}(t^+) + \tilde{U}_i^{(p)}(t^+) - \tilde{U}_i^{(p)}(t^-) + \tilde{U}_i^{(p)}(t^-) - U_i^{(p)}(t^-) \\ &\equiv e_+ + e_0 + e_-. \end{aligned}$$

By Lemma 5.3 (or Lemma 5.4),  $U$  is an interpolant of  $\tilde{U}$  and so, by Lemma 5.1, we have

$$|e_+| \leq C k_{ij}^{r+1-p} \|\tilde{U}_i^{(r+1)}\|_{L_\infty(I_{ij})} + C \sum_{x \in \mathcal{N}_{ij}} \sum_{m=1}^r k_{ij}^{m-p} |[\tilde{U}_i^{(m)}]_x|$$

for  $p = 0, \dots, r+1$  and  $r = 0, \dots, \bar{q}$ . Note that the second sum starts at  $m = 1$  rather than at  $m = 0$ , since  $\tilde{U}$  is continuous. Similarly, we have

$$|e_-| \leq C k_{i,j-1}^{r+1-p} \|\tilde{U}_i^{(r+1)}\|_{L_\infty(I_{i,j-1})} + C \sum_{x \in \mathcal{N}_{i,j-1}} \sum_{m=1}^r k_{i,j-1}^{m-p} |[\tilde{U}_i^{(m)}]_x|.$$

To estimate  $e_0$ , we note that  $e_0 = 0$  for  $p = 0$ , since  $\tilde{U}$  is continuous. For  $p = 1, \dots, \bar{q} + 1$ , Lemma 5.7 gives  $|e_0| = |[\tilde{U}_i^{(p)}]_t| \leq C \sum_{n=0}^{p-1} C_{U,p-1}^{p-n} \| [U^{(n)}]_t \|_{l_\infty}$ . By assumption (A2), it then follows that (5.16) holds for  $r = 0$ .

Assume now that (5.16) holds for  $r = \bar{r} - 1 \geq 0$ . Then, by Lemma 5.7 and assumption (A5'), it follows that

$$\begin{aligned} |e_+| &\leq C k_{ij}^{\bar{r}+1-p} C_{U,\bar{r}}^{\bar{r}+1} + C \sum_{x \in \mathcal{N}_{ij}} \sum_{m=1}^{\bar{r}} k_{ij}^{m-p} \sum_{n=0}^{m-1} C_{U,m-1}^{m-n} \| [U^n]_x \|_{l_\infty} \\ &\leq C k_{ij}^{\bar{r}+1-p} C_{U,\bar{r}}^{\bar{r}+1} + C \sum_{m=1}^{\bar{r}} k_{ij}^{m-p} C_{U,m-1}^{m-n} k_{ij}^{(\bar{r}-1)+1-n} C_{U,\bar{r}-1}^{(\bar{r}-1)+1} \\ &\leq C k_{ij}^{\bar{r}+1-p} C_{U,\bar{r}}^{\bar{r}+1} \left( 1 + \sum_{m=1}^{\bar{r}} (k_{ij} C_{U,\bar{r}-1})^{m-1-n} \right) \leq C k_{ij}^{\bar{r}+1-p} C_{U,\bar{r}}^{\bar{r}+1}. \end{aligned}$$

Similarly, we obtain the estimate  $|e_-| \leq C k_{ij}^{\bar{r}+1-p} C_{U,\bar{r}}^{\bar{r}+1}$ . Finally, we use Lemma 5.7 and assumption (A5') to obtain the estimate

$$\begin{aligned} |e_0| &\leq C \sum_{n=0}^{p-1} C_{U,p-1}^{p-n} \| [U^n]_t \|_{l_\infty} \leq C \sum_{n=0}^{p-1} C_{U,p-1}^{p-n} k_{ij}^{(\bar{r}-1)+1-n} C_{U,\bar{r}-1}^{(\bar{r}-1)+1} \\ &= C k_{ij}^{\bar{r}+1-p} C_{U,\bar{r}}^{\bar{r}+1} \sum_{n=0}^{p-1} (k_{ij} C_{U,\bar{r}})^{p-1-n} \leq C k_{ij}^{\bar{r}+1-p} C_{U,\bar{r}}^{\bar{r}+1}. \end{aligned}$$

Summing up, we thus obtain  $|[U_i^{(p)}]_t| \leq |e_+| + |e_0| + |e_-| \leq C k_{ij}^{\bar{r}+1-p} C_{U,\bar{r}}^{\bar{r}+1}$ , and so (5.16) follows by induction.  $\square$

By Lemmas 5.7 and 5.8, we now obtain the following estimate for derivatives of the solution  $U$ .

LEMMA 5.9 (derivative estimates for  $U$ ). *Let  $U$  be the mcG( $q$ ) or mdG( $q$ ) solution of (1.1). If assumptions (A1)–(A5) hold, then there is a constant  $C = C(\bar{q}, c_k, \alpha) > 0$  such that*

$$(5.17) \quad \|U^{(p)}\|_{L_\infty(\mathcal{T}, l_\infty)} \leq C C_f^p, \quad p = 1, \dots, \bar{q}.$$

*Proof.* By Lemma 5.3 (or Lemma 5.4),  $U$  is an interpolant of  $\tilde{U}$  and so, by Lemma 5.1, we have

$$\|U_i^{(p)}\|_{L_\infty(I_{ij})} = \|(\pi \tilde{U}_i)^{(p)}\|_{L_\infty(I_{ij})} \leq C' \|\tilde{U}_i^{(p)}\|_{L_\infty(I_{ij})} + C' \sum_{x \in \mathcal{N}_{ij}} \sum_{m=1}^{p-1} k_{ij}^{m-p} |[\tilde{U}_i^{(m)}]_x|$$

for some constant  $C' = C'(\bar{q})$ . For  $p = 1$ , we thus obtain the estimate

$$\|\dot{U}_i\|_{L_\infty(I_{ij})} \leq C' \|\dot{\tilde{U}}_i\|_{L_\infty(I_{ij})} = C' \|f_i(U)\|_{L_\infty(I_{ij})} \leq C' C_f$$

by assumption (A4), and so (5.17) holds for  $p = 1$ .

For  $p = 2, \dots, \bar{q}$ , assuming that (A5') holds for  $C_{U,p-1}$ , we use Lemmas 5.7 and 5.8 (with  $r = p - 1$ ) and assumption (A2) to obtain

$$\begin{aligned} \|U_i^{(p)}\|_{L_\infty(I_{ij})} &\leq CC_{U,p-1}^p + C \sum_{x \in \mathcal{N}_{ij}} \sum_{m=1}^{p-1} k_{ij}^{m-p} \sum_{n=0}^{m-1} C_{U,m-1}^{m-n} \| [U^{(n)}]_x \|_{l_\infty} \\ &\leq CC_{U,p-1}^p + C \sum_{m=1}^{p-1} k_{ij}^{m-p} C_{U,m-1}^{m-n} k_{ij}^{(p-1)+1-n} C_{U,p-1}^{(p-1)+1} \\ &\leq CC_{U,p-1}^p \left( 1 + \sum (k_{ij} C_{U,m-1})^{m-n} \right) \leq CC_{U,p-1}^p, \end{aligned}$$

where  $C = C(\bar{q}, c_k, c'_k, \alpha)$ . This holds for all components  $i$  and all local intervals  $I_{ij}$  within the time slab  $\mathcal{T}$ , and so

$$\|U^{(p)}\|_{L_\infty(\mathcal{T}, l_\infty)} \leq CC_{U,p-1}^p, \quad p = 1, \dots, \bar{q},$$

where by definition  $C_{U,p-1}$  is a constant such that  $\|U^{(n)}\|_{L_\infty(\mathcal{T}, l_\infty)} \leq C_{U,p-1}^n$  for  $n = 1, \dots, p - 1$ . Starting at  $p = 1$ , we now define  $C_{U,1} = C_1 C_f$  with  $C_1 = C' = C'(\bar{q})$ . It then follows that (A5') holds for  $C_{U,1}$  with  $c'_k = C' c_k$ , and thus

$$\|U^{(2)}\|_{L_\infty(\mathcal{T}, l_\infty)} \leq CC_{U,2-1}^2 = CC_{U,1}^2 \equiv C_2 C_f^2,$$

where  $C_2 = C_2(\bar{q}, c_k, \alpha)$ . We may thus define  $C_{U,2} = \max(C_1 C_f, \sqrt{C_2} C_f)$ . Continuing, we note that (A5') holds for  $C_{U,2}$ , and thus

$$\|U^{(3)}\|_{L_\infty(\mathcal{T}, l_\infty)} \leq CC_{U,3-1}^3 = CC_{U,2}^3 \equiv C_3 C_f^3,$$

where  $C_3 = C_3(\bar{q}, c_k, \alpha)$ . In this way, we obtain a sequence of constants  $C_1, \dots, C_{\bar{q}}$ , depending only on  $\bar{q}, c_k$ , and  $\alpha$ , such that  $\|U^{(p)}\|_{L_\infty(\mathcal{T}, l_\infty)} \leq C_p C_f^p$  for  $p = 1, \dots, \bar{q}$ , and so (5.17) follows if we take  $C = \max_{i=1, \dots, \bar{q}} C_i$ .  $\square$

Having now removed the additional assumption (A5'), we obtain the following version of Lemma 5.8.

LEMMA 5.10 (jump estimates for  $U$ ). *Let  $U$  be the mcG( $q$ ) or mdG( $q$ ) solution of (1.1). If assumptions (A1)–(A5) hold, then there is a constant  $C = C(\bar{q}, c_k, \alpha) > 0$  such that*

$$(5.18) \quad \|[U^{(p)}]_{t_{i,j-1}}\|_{l_\infty} \leq C k_{ij}^{\bar{q}+1-p} C_f^{\bar{q}+1}, \quad p = 0, \dots, \bar{q},$$

for each local interval  $I_{ij}$ , where  $t_{i,j-1}$  is an internal node of the time slab  $\mathcal{T}$ .

Similarly, we obtain estimates for the discrete dual solution  $\Phi$  and the function  $\varphi$ . In Lemma 5.11, we present the estimates for the function  $\varphi$ .

LEMMA 5.11 (estimates for  $\varphi$ ). *Let  $\varphi$  be defined as in (5.6). If assumptions (A1)–(A5) hold, then there is a constant  $C = C(\bar{q}, c_k, \alpha) > 0$  such that*

$$(5.19) \quad \|\varphi_i^{(p)}\|_{L_\infty(I_{ij})} \leq CC_f^{p+1} \|\Phi\|_{L_\infty(\mathcal{T}, l_\infty)}, \quad p = 0, \dots, q_{ij},$$

and

$$(5.20) \quad |[\varphi_i^{(p)}]_x| \leq C k_{ij}^{r_{ij}-p} C_f^{r_{ij}+1} \|\Phi\|_{L_\infty(\mathcal{T}, l_\infty)} \quad \forall x \in \mathcal{N}_{ij}, \quad p = 0, \dots, q_{ij} - 1,$$

with  $r_{ij} = q_{ij}$  for the mcG( $q$ ) method and  $r_{ij} = q_{ij} + 1$  for the mdG( $q$ ) method. This holds for each local interval  $I_{ij}$  within the time slab  $\mathcal{T}$ .

**5.3.4. Interpolation estimates.** Using the basic interpolation estimate of section 5.2, we now obtain the following important interpolation estimates for the function  $\varphi$ .

LEMMA 5.12 (interpolation estimates for  $\varphi$ ). *Let  $\varphi$  be defined as in (5.6). If assumptions (A1)–(A5) hold, then there is a constant  $C = C(\bar{q}, c_k, \alpha) > 0$  such that*

$$(5.21) \quad \|\pi_{\text{cG}}^{[q_{ij}-2]} \varphi_i - \varphi_i\|_{L_\infty(I_{ij})} \leq C k_{ij}^{q_{ij}-1} C_f^{q_{ij}} \|\Phi\|_{L_\infty(\mathcal{T}, l_\infty)}, \quad q_{ij} = \bar{q} \geq 2,$$

and

$$(5.22) \quad \|\pi_{\text{dG}}^{[q_{ij}-1]} \varphi_i - \varphi_i\|_{L_\infty(I_{ij})} \leq C k_{ij}^{q_{ij}} C_f^{q_{ij}+1} \|\Phi\|_{L_\infty(\mathcal{T}, l_\infty)}, \quad q_{ij} = \bar{q} \geq 1,$$

for each local interval  $I_{ij}$  within the time slab  $\mathcal{T}$ .

*Proof.* To prove (5.21), we use Lemma 5.1, with  $r = q_{ij} - 2$  and  $p = 0$ , together with Lemma 5.11, to obtain

$$\begin{aligned} \|\pi_{\text{cG}}^{[q_{ij}-2]} \varphi_i - \varphi_i\|_{L_\infty(I_{ij})} &\leq C k_{ij}^{q_{ij}-1} \|\varphi_i^{(q_{ij}-1)}\|_{L_\infty(I_{ij})} + C \sum_{x \in \mathcal{N}_{ij}} \sum_{m=0}^{q_{ij}-2} k_{ij}^m |[\varphi_i^{(m)}]_x| \\ &\leq C k_{ij}^{q_{ij}-1} C_f^{q_{ij}} \|\Phi\|_{L_\infty(\mathcal{T}, l_\infty)} + C \sum_{x \in \mathcal{N}_{ij}} \sum_{m=0}^{q_{ij}-2} k_{ij}^m k_{ij}^{q_{ij}-m} C_f^{q_{ij}+1} \|\Phi\|_{L_\infty(\mathcal{T}, l_\infty)} \\ &= C k_{ij}^{q_{ij}-1} C_f^{q_{ij}} \|\Phi\|_{L_\infty(\mathcal{T}, l_\infty)} + C k_{ij}^{q_{ij}} C_f^{q_{ij}+1} \|\Phi\|_{L_\infty(\mathcal{T}, l_\infty)}, \end{aligned}$$

from which the estimate follows. The estimate for  $\pi_{\text{dG}}^{[q_{ij}-1]} \varphi_i - \varphi_i$  is obtained similarly.  $\square$

REMARK 5.1. *Note that there is only a weak dependence on  $c_k$  and  $\alpha$ , since the jump term contains an extra factor  $k_{ij}$ . If higher-order terms can be ignored, then the dependence on  $c_k$  and  $\alpha$  can be removed.*

**6. A priori error estimates.** To prove a priori error estimates for the mcG( $q$ ) and mdG( $q$ ) methods, we derive error representations in section 6.1 and then obtain the a priori error estimates in section 6.2 for the general nonlinear case. We refer to [34] for a sharp a priori error estimate in the case of a parabolic model problem.

**6.1. Error representation.** For each of the two methods, mcG( $q$ ) and mdG( $q$ ), we represent the error in terms of the discrete dual solution  $\Phi$  and an interpolant  $\pi u$  of the exact solution  $u$  of (1.1), using the special interpolants  $\pi u = \pi_{\text{cG}}^{[q]} u$  or  $\pi u = \pi_{\text{dG}}^{[q]} u$  defined in section 5.

We write the error  $e = U - u$  in the form

$$(6.1) \quad e = \bar{e} + (\pi u - u),$$

where  $\bar{e} \equiv U - \pi u$  is represented in terms of the discrete dual solution and the residual of the interpolant. An estimate for the second part of the error,  $\pi u - u$ , follows directly from an interpolation estimate.

In Lemma 6.1, we present the error representation for the mcG( $q$ ) method, and then present the corresponding representation for the mdG( $q$ ) method in Lemma 6.2. The error representations are obtained directly by choosing  $\bar{e}$  as a test function for the discrete dual problems (2.12) and (2.15).

LEMMA 6.1 (error representation for mcG( $q$ )). *Let  $U$  be the mcG( $q$ ) solution of (1.1), let  $\Phi$  be the corresponding mcG( $q$ )\* solution of the dual problem (2.9), and let*

$\pi u$  be any trial space approximation of the exact solution  $u$  of (1.1) that interpolates  $u$  at the end-points of every local interval. Then

$$L_{\psi,g}(\bar{e}) \equiv (\bar{e}(T), \psi) + \int_0^T (\bar{e}, g) dt = - \int_0^T (R(\pi u, \cdot), \Phi) dt,$$

where  $\bar{e} \equiv U - \pi u$ .

LEMMA 6.2 (error representation for mdG( $q$ )). *Let  $U$  be the mdG( $q$ ) solution of (1.1), let  $\Phi$  be the corresponding mdG( $q$ )<sup>\*</sup> solution of the dual problem (2.9), and let  $\pi u$  be any trial space approximation of the exact solution  $u$  of (1.1) that interpolates  $u$  at the right end-point of every local interval. Then*

$$L_{\psi,g}(\bar{e}) = - \sum_{i=1}^N \sum_{j=1}^{M_i} \left[ [\pi u_i]_{i,j-1} \Phi_i(t_{i,j-1}^+) + \int_{I_{ij}} R_i(\pi u, \cdot) \Phi_i dt \right],$$

where  $\bar{e} \equiv U - \pi u$ .

With a special choice of interpolant,  $\pi u = \pi_{cG}^{[q]} u$  and  $\pi u = \pi_{dG}^{[q]} u$ , respectively, we obtain the following versions of the error representations.

COROLLARY 6.3 (error representation for mcG( $q$ )). *Let  $U$  be the mcG( $q$ ) solution of (1.1) and let  $\Phi$  be the corresponding mcG( $q$ )<sup>\*</sup> solution of the dual problem (2.9). Then*

$$L_{\psi,g}(\bar{e}) = \int_0^T (f(\pi_{cG}^{[q]} u, \cdot) - f(u, \cdot), \Phi) dt.$$

*Proof.* Integrate by parts and use the definition of the interpolant  $\pi_{cG}^{[q]}$ . □

COROLLARY 6.4 (error representation for mdG( $q$ )). *Let  $U$  be the mdG( $q$ ) solution of (1.1) and let  $\Phi$  be the corresponding mdG( $q$ )<sup>\*</sup> solution of the dual problem (2.9). Then*

$$L_{\psi,g}(\bar{e}) = \int_0^T (f(\pi_{dG}^{[q]} u, \cdot) - f(u, \cdot), \Phi) dt.$$

*Proof.* Integrate by parts and use the definition of the interpolant  $\pi_{dG}^{[q]}$ . □

**6.2. A priori error estimates for the general nonlinear problem.** Using the error representations of section 6.1, the stability estimates of section 4, and the interpolation estimates of section 5, we now prove our main results: a priori error estimates for general order mcG( $q$ ) and mdG( $q$ ).

THEOREM 6.5 (a priori error estimate for mcG( $q$ )). *Let  $U$  be the mcG( $q$ ) solution of (1.1) and let  $\Phi$  be the corresponding mcG( $q$ )<sup>\*</sup> solution of the dual problem (2.9). Then there is a constant  $C = C(q) > 0$  such that*

$$(6.2) \quad |L_{\psi,g}(\bar{e})| \leq CS(T) \|k^{q+1} \bar{u}^{(q+1)}\|_{L_\infty([0,T],l_2)},$$

where  $(k^{q+1} \bar{u}^{(q+1)})_i(t) = k_{ij}^{q_{ij}+1} \|u_i^{(q_{ij}+1)}\|_{L_\infty(I_{ij})}$  for  $t \in I_{ij}$ , and where the stability factor  $S(T)$  is given by  $S(T) = \int_0^T \|J^\top(\pi_{cG}^{[q]} u, u, \cdot) \Phi\|_{l_2} dt$ . Furthermore, if assumptions (A1)–(A5) hold, then there is a constant  $C = C(q, c_k, \alpha) > 0$  such that

$$(6.3) \quad |L_{\psi,g}(\bar{e})| \leq C\bar{S}(T) \|k^{2q} \bar{u}^{(2q)}\|_{L_\infty([0,T],l_1)},$$

where  $(k^{2q}\bar{u}^{(2q)})_i(t) = k_{ij}^{2q_{ij}} C_f^{q_{ij}-1} \|u_i^{(q_{ij}+1)}\|_{L_\infty(I_{ij})}$  for  $t \in I_{ij}$ , and where the stability factor  $\bar{S}(T)$  is given by

$$\bar{S}(T) = \int_0^T C_f \|\Phi\|_{L_\infty(\mathcal{T}, l_\infty)} dt = \sum_{n=1}^M K_n C_f \|\Phi\|_{L_\infty(\mathcal{T}_n, l_\infty)}.$$

*Proof.* By Corollary 6.3, we obtain

$$L_{\psi,g}(\bar{e}) = \int_0^T (f(\pi_{cG}^{[q]}u, \cdot) - f(u, \cdot), \Phi) dt = \int_0^T (\pi_{cG}^{[q]}u - u, J^\top(\pi_{cG}^{[q]}u, u, \cdot)\Phi) dt.$$

By Lemma 5.1, it now follows that

$$|L_{\psi,g}(\bar{e})| \leq C \|k^{q+1}\bar{u}^{q+1}\|_{L_\infty([0,T], l_2)} \int_0^T \|J^\top(\pi_{cG}^{[q]}u, u, \cdot)\Phi\|_{l_2} dt,$$

which proves (6.2). To prove (6.3), we note that by definition,  $\pi_{cG}^{[q_{ij}]}u_i - u_i$  is orthogonal to  $\mathcal{P}^{q_{ij}-2}(I_{ij})$  for each local interval  $I_{ij}$ , and so, recalling that  $\varphi = J^\top(\pi_{cG}^{[q]}u, u, \cdot)\Phi$ ,

$$L_{\psi,g}(\bar{e}) = \sum_{i,j} \int_{I_{ij}} (\pi_{cG}^{[q_{ij}]}u_i - u_i)\varphi_i dt = \sum_{i,j} \int_{I_{ij}} (\pi_{cG}^{[q_{ij}]}u_i - u_i)(\varphi_i - \pi_{cG}^{[q_{ij}-2]}\varphi_i) dt,$$

where we take  $\pi_{cG}^{[q_{ij}-2]}\varphi_i \equiv 0$  for  $q_{ij} = 1$ . By Lemmas 5.1 and 5.12, it now follows that

$$\begin{aligned} |L_{\psi,g}(\bar{e})| &\leq \int_0^T |(\pi_{cG}^{[q]}u - u, \varphi - \pi_{cG}^{[q-2]}\varphi)| dt \\ &= \int_0^T |(k^{q-1}C_f^{q-1}(\pi_{cG}^{[q]}u - u), k^{-(q-1)}C_f^{-(q-1)}(\varphi - \pi_{cG}^{[q-2]}\varphi))| dt \\ &\leq C \|k^{2q}\bar{u}^{(2q)}\|_{L_\infty([0,T], l_1)} \int_0^T C_f \|\Phi\|_{L_\infty(\mathcal{T}, l_\infty)} dt \\ &= C\bar{S}(T) \|k^{2q}\bar{u}^{(2q)}\|_{L_\infty([0,T], l_1)}, \end{aligned}$$

where  $\bar{S}(T) = \int_0^T C_f \|\Phi\|_{L_\infty(\mathcal{T}, l_\infty)} dt = \sum_{n=1}^M K_n C_f \|\Phi\|_{L_\infty(\mathcal{T}_n, l_\infty)}$ .  $\square$

Similarly, we obtain the following a priori error estimate for the mdG( $q$ ) method.

**THEOREM 6.6** (a priori error estimate for mdG( $q$ )). *Let  $U$  be the mdG( $q$ ) solution of (1.1) and let  $\Phi$  be the corresponding mdG( $q$ )<sup>\*</sup> solution of the dual problem (2.9). Then there is a constant  $C = C(q) > 0$  such that*

$$(6.4) \quad |L_{\psi,g}(\bar{e})| \leq CS(T) \|k^{q+1}\bar{u}^{(q+1)}\|_{L_\infty([0,T], l_2)},$$

where  $(k^{q+1}\bar{u}^{(q+1)})_i(t) = k_{ij}^{q_{ij}+1} \|u_i^{(q_{ij}+1)}\|_{L_\infty(I_{ij})}$  for  $t \in I_{ij}$ , and where the stability factor  $S(T)$  is given by  $S(T) = \int_0^T \|J^\top(\pi_{dG}^{[q]}u, u, \cdot)\Phi\|_{l_2} dt$ . Furthermore, if assumptions (A1)–(A5) hold, then there is a constant  $C = C(q, c_k, \alpha) > 0$  such that

$$(6.5) \quad |L_{\psi,g}(\bar{e})| \leq C\bar{S}(T) \|k^{2q+1}\bar{u}^{(2q+1)}\|_{L_\infty([0,T], l_1)},$$

where  $(k^{2q+1}\bar{u}^{(2q+1)})_i(t) = k_{ij}^{2q_{ij}+1} C_f^{q_{ij}} \|u_i^{(q_{ij}+1)}\|_{L_\infty(I_{ij})}$  for  $t \in I_{ij}$ , and where the stability factor  $\bar{S}(T)$  is given by

$$\bar{S}(T) = \int_0^T C_f \|\Phi\|_{L_\infty(\mathcal{T}, l_\infty)} dt = \sum_{n=1}^M K_n C_f \|\Phi\|_{L_\infty(\mathcal{T}_n, l_\infty)}.$$

Using the stability estimate proved in section 4, we obtain the following bound for the stability factor  $\bar{S}(T)$ .

LEMMA 6.7. *Assume that  $K_n C_q C_f \leq 1$  for all time slabs  $\mathcal{T}_n$ , with  $C_q > 0$  the constant in Theorem 4.2, and take  $g = 0$  in (2.9). Then*

$$(6.6) \quad \bar{S}(T) \leq \|\psi\|_{l_\infty} e^{C_q \bar{C}_f T},$$

where  $\bar{C}_f = \max_{[0,T]} C_f$ .

*Proof.* By Theorem 4.2, we obtain

$$\|\Phi\|_{L_\infty(\mathcal{T}_n, l_\infty)} \leq C_q \|\psi\|_{l_\infty} \exp\left(\sum_{m=n+1}^M K_m C_q C_f\right) \leq C_q \|\psi\|_{l_\infty} e^{C_q \bar{C}_f (T-T_n)},$$

and so

$$\begin{aligned} \bar{S}(T) &= \sum_{n=1}^M K_n C_f \|\Phi\|_{L_\infty(\mathcal{T}_n, l_\infty)} dt \leq \|\psi\|_{l_\infty} \sum_{n=1}^M K_n C_q \bar{C}_f e^{C_q \bar{C}_f (T-T_n)} \\ &\leq \|\psi\|_{l_\infty} \int_0^T C_q \bar{C}_f e^{C_q \bar{C}_f t} dt \leq \|\psi\|_{l_\infty} e^{C_q \bar{C}_f T}. \quad \square \end{aligned}$$

Finally, we rewrite the estimates of Theorems 6.5 and 6.6 for special choices of data  $\psi$  and  $g$ . We first take  $\psi = 0$ . With  $g_n = 0$  for  $n \neq i$ ,  $g_i(t) = 0$  for  $t \notin I_{ij}$ , and

$$g_i(t) = \text{sgn}(\bar{e}_i(t))/k_{ij}, \quad t \in I_{ij},$$

we obtain  $L_{\psi,g}(\bar{e}) = \frac{1}{k_{ij}} \int_{I_{ij}} |\bar{e}_i(t)| dt$  and so  $\|\bar{e}_i\|_{L_\infty(I_{ij})} \leq CL_{\psi,g}(\bar{e})$  by an inverse estimate. By definition, it follows that  $\|e_i\|_{L_\infty(I_{ij})} \leq CL_{\psi,g}(\bar{e}) + Ck_{ij}^{q_{ij}+1} \|u_i^{q_{ij}+1}\|_{L_\infty(I_{ij})}$ . Note that for this choice of  $g$ , we have  $\|g\|_{L_1([0,T],l_2)} = \|g\|_{L_1([0,T],l_\infty)} = 1$ .

We also make the choice  $g = 0$ . Noting that  $\bar{e}(T) = e(T)$ , since  $\pi u(T) = u(T)$ , we obtain

$$L_{\psi,g}(\bar{e}) = (e(T), \psi) = |e_i(T)|$$

for  $\psi_i = \text{sgn}(e_i(T))$  and  $\psi_n = 0$  for  $n \neq i$ , and

$$L_{\psi,g}(\bar{e}) = (e(T), \psi) = \|e(T)\|_{l_2}$$

for  $\psi = e(T)/\|e(T)\|_{l_2}$ . Note that for both choices of  $\psi$ , we have  $\|\psi\|_{l_\infty} \leq 1$ .

With these choices of data, we obtain the following versions of the a priori error estimates.

COROLLARY 6.8 (a priori error estimate for mcG( $q$ )). *Let  $U$  be the mcG( $q$ ) solution of (1.1). Then there is a constant  $C = C(q) > 0$  such that*

$$(6.7) \quad \|e\|_{L_\infty([0,T],l_\infty)} \leq CS(T) \|k^{q+1} \bar{u}^{(q+1)}\|_{L_\infty([0,T],l_2)},$$

where the stability factor  $S(T) = \int_0^T \|J^\top(\pi_{\text{cG}}^{[q]} u, u, \cdot)\Phi\|_{l_2} dt$  is taken as the maximum over  $\psi = 0$  and  $\|g\|_{L_1([0,T],l_\infty)} = 1$ . Furthermore, if assumptions (A1)–(A5) and the assumptions of Lemma 6.7 hold, then there is a constant  $C = C(q, c_k, \alpha)$  such that

$$(6.8) \quad \|e(T)\|_{l_p} \leq C \bar{S}(T) \|k^{2q} \bar{u}^{(2q)}\|_{L_\infty([0,T],l_1)}$$

for  $p = 2, \infty$ , where the stability factor  $\bar{S}(T)$  is given by  $\bar{S}(T) = e^{C_q \bar{C}_f T}$ .

COROLLARY 6.9 (a priori error estimate for mdG( $q$ )). Let  $U$  be the mdG( $q$ ) solution of (1.1). Then there is a constant  $C = C(q) > 0$  such that

$$(6.9) \quad \|e\|_{L_\infty([0,T],l_\infty)} \leq CS(T) \|k^{q+1} \bar{u}^{(q+1)}\|_{L_\infty([0,T],l_2)},$$

where the stability factor  $S(T) = \int_0^T \|J^\top(\pi_{\text{dG}}^{[q]} u, u, \cdot) \Phi\|_{l_2} dt$  is taken as the maximum over  $\psi = 0$  and  $\|g\|_{L_1([0,T],l_\infty)} = 1$ . Furthermore, if assumptions (A1)–(A5) and the assumptions of Lemma 6.7 hold, then there is a constant  $C = C(q, c_k, \alpha)$  such that

$$(6.10) \quad \|e(T)\|_{l_p} \leq C \bar{S}(T) \|k^{2q+1} \bar{u}^{(2q+1)}\|_{L_\infty([0,T],l_1)}$$

for  $p = 2, \infty$ , where the stability factor  $\bar{S}(T)$  is given by  $\bar{S}(T) = e^{C_q \bar{C}_f T}$ .

The stability factor  $S(T)$  that appears in the a priori error estimates is obtained from the discrete solution  $\Phi$  of the dual problem (4.1), and can thus be computed by solving the discrete dual problem. Numerical computation of the stability factor reveals the exact nature of the problem, in particular, whether or not the problem is parabolic; if the stability factor is of unit size and does not grow, then the problem is parabolic by definition; see [36].

**6.3. A note on quadrature errors.** The error representations presented in section 6.1 are based on the Galerkin orthogonalities of the mcG( $q$ ) and mdG( $q$ ) methods. In particular, for the mcG( $q$ ) method, we assume that

$$\int_0^T (R(U, \cdot), \Phi) dt = 0.$$

In the presence of quadrature errors, this term is nonzero. As a result, we obtain an additional term of the form

$$\int_0^T (\tilde{f}(U, \cdot) - f(U, \cdot), \Phi) dt,$$

where  $\tilde{f}$  is the interpolant of  $f$  corresponding the quadrature rule that is used. A convenient choice of quadrature for the mcG( $q$ ) method is Lobatto quadrature with  $q + 1$  nodal points [32], which means that the quadrature error is of order  $2(q+1) - 2 = 2q$  and so (super)convergence of order  $2q$  is obtained also in the presence of quadrature errors. Similarly for the mdG( $q$ ) method, we use Radau quadrature with  $q + 1$  nodal points, which means that the quadrature error is of order  $2(q + 1) - 1 = 2q + 1$ , and so the  $2q + 1$  convergence order of mdG( $q$ ) is also maintained under quadrature.

**7. A numerical example.** We conclude by demonstrating the convergence of the multiadaptive methods in the case of a simple test problem.

Consider the problem

$$(7.1) \quad \begin{aligned} \dot{u}_1 &= u_2, \\ \dot{u}_2 &= -u_1, \\ \dot{u}_3 &= -u_2 + 2u_4, \\ \dot{u}_4 &= u_1 - 2u_3, \\ \dot{u}_5 &= -u_2 - 2u_4 + 4u_6, \\ \dot{u}_6 &= u_1 + 2u_3 - 4u_5 \end{aligned}$$

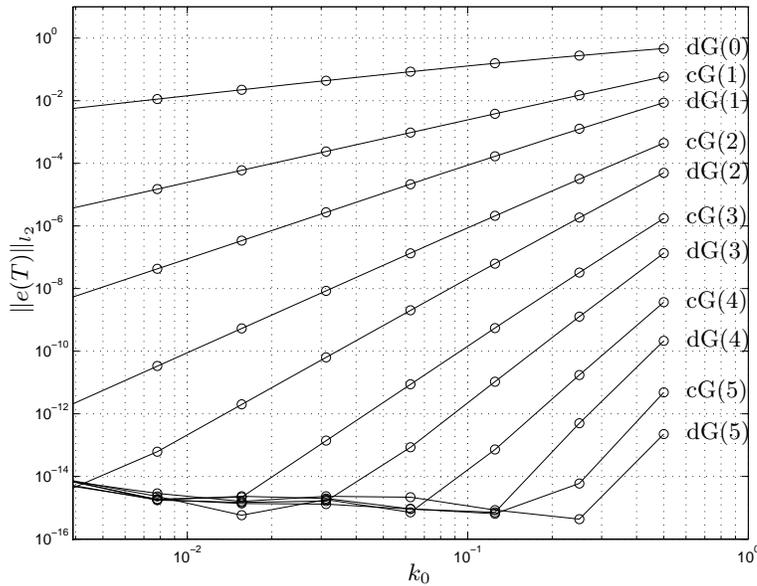


FIG. 6. Convergence of the error at final time for the solution of the test problem (7.1) with mcG( $q$ ) and mdG( $q$ ),  $q \leq 5$ .

TABLE 1  
Order of convergence  $p$  for mcG( $q$ ).

mcG( $q$ )	1	2	3	4	5
$p$	1.99	3.96	5.92	7.82	9.67
$2q$	2	4	6	8	10

TABLE 2  
Order of convergence  $p$  for mdG( $q$ ).

mdG( $q$ )	0	1	2	3	4	5
$p$	0.92	2.96	4.94	6.87	9.10	—
$2q + 1$	1	3	5	7	9	11

on  $[0, 1]$  with initial condition  $u(0) = (0, 1, 0, 2, 0, 3)$ . The solution is given by  $u(t) = (\sin t, \cos t, \sin t + \sin 2t, \cos t + \cos 2t, \sin t + \sin 2t + \sin 4t, \cos t + \cos 2t + \cos 4t)$ . For given  $k_0 > 0$ , we take  $k_i(t) = k_0$  for  $i = 1, 2$ ,  $k_i(t) = k_0/2$  for  $i = 3, 4$ , and  $k_i(t) = k_0/4$  for  $i = 5, 6$ , and study the convergence of the error  $\|e(T)\|_{l_2}$  with decreasing  $k_0$ . From the results presented in Figure 6 and Tables 1 and 2, it is clear that the predicted order of convergence is obtained.

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