3 Common and unusual finite elements

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This chapter provides a glimpse of the considerable range of finite elements in the literature. Many of the elements presented here are implemented as part of the FEniCS Project already; some are future work. The universe of finite elements extends far beyond what we consider here. In particular, we consider only simplicial, polynomial-based elements. We thus bypass elements defined on quadrilaterals and hexahedra, composite and macro-element techniques, as well as XFEM-type methods. Even among polynomial-based elements on simplices, the list of elements can be extended. Nonetheless, this chapter presents a comprehensive collection of some of the most common, and some more unusual, finite elements.

3.1 The finite element definition

The Ciarlet definition of a *finite element* was first introduced in a set of lecture notes by Ciarlet (1975) and became popular after his 1978 book (Ciarlet, 2002). It remains the standard definition today, see for example Brenner and Scott (2008). The definition, which was also presented in Chapter 2, reads as follows:

Definition 3.1 (Finite element (Ciarlet, 2002)) *A finite element is defined by a triple* $(T, \mathcal{V}, \mathcal{L})$, *where*

- *the domain* T *is a bounded, closed subset of* $\mathbb{R}^d$ *(for* $d = 1, 2, 3, \dots$*) with nonempty interior and piecewise smooth boundary;*
- *the space* $\mathcal{V} = \mathcal{V}(T)$ *is a finite dimensional function space on* T *of dimension* n*;*
- *the set of degrees of freedom (nodes)* $\mathcal{L} = \{\ell_1, \ell_2, \dots, \ell_n\}$ *is a basis for the dual space* $\mathcal{V}'$*; that is, the space of bounded linear functionals on* $\mathcal{V}$.

Similar ideas were introduced earlier in Ciarlet and Raviart (1972)[1], in which unisolvence[2] of a set of interpolation points $\{x^i\}_i$ was discussed. This is closely related to the unisolvence of $\mathcal{L}$ when the degrees of freedom are given by by $\ell_i(v) = v(x^i)$. Conditions for uniquely determining a polynomial based on interpolation of function values and derivatives at a set of points was also discussed in Bramble and Zlámal (1970), although the term unisolvence was not used.

For any finite element, one may define a local basis for $\mathcal{V}$ that is dual to the degrees of freedom. Such a basis $\{\phi_1^T, \phi_2^T \dots, \phi_n^T\}$ satisfies $\ell_i(\phi_j^T) = \delta_{ij}$ for $1 \leqslant i, j \leqslant n$ and is called the *nodal basis*. It is typically this basis that is used in finite element computations.

[1] The Ciarlet triple was originally written as (K, P, Σ) with K denoting T, P denoting $\mathcal{V}$, and Σ denoting $\mathcal{L}$.

[2] To check whether a given set of linear functionals is a basis for $\mathcal{V}'$, one may check whether it is *unisolvent* for $\mathcal{V}$; that is, for $v \in \mathcal{V}$, $\ell_i(v) = 0$ for $i = 1, \dots, n$ if and only if $v = 0$.

Also associated with a finite element is a *local interpolation operator*, sometimes called a *nodal interpolant*. Given some function f on T, the nodal interpolant is defined by

$$\Pi_T(f) = \sum_{i=1}^{n} \ell_i(f)\phi_i^T, \tag{3.1}$$

assuming that f is smooth enough for all of the degrees of freedom acting on it to be well-defined.

Once a local finite element space is defined, it is relatively straightforward to define a global finite element space over a tessellation $\mathcal{T}_h$. One defines the global space to consist of functions whose restrictions to each $T \in \mathcal{T}_h$ lie in the local space $\mathcal{V}(T)$ and that also satisfy any required continuity requirements. Typically, the degrees of freedom for each local element are chosen such that if the degrees of freedom on a common interface between two adjacent cells T and T' agree, then a function will satisfy the required continuity condition.

When constructing a global finite element space, it is common to construct a single *reference finite element* $(\hat{T}, \hat{\mathcal{V}}, \hat{\mathcal{L}})$ and map it to each cell in the mesh. As we are dealing with a simplicial geometry, the mapping between $\hat{T}$ and each $T \in \mathcal{T}_h$ will be affine. Originally defined for the purpose of error estimation, but also useful for computation, is the notion of *affine equivalence*. Let $F_T : \hat{T} \to T$ denote this affine map. Let $v \in \mathcal{V}$. The *pullback* associated with the affine map is given by $\mathcal{F}^*(v)(\hat{x}) = v(F_T(\hat{x}))$ for all $\hat{x} \in \hat{T}$. Given a functional $\hat{\ell} \in \hat{\mathcal{V}}'$, its *pushforward* acts on a function in $v \in \mathcal{V}$ by $\mathcal{F}_*(\hat{\ell})(v) = \hat{\ell}(\mathcal{F}^*(v))$.

Definition 3.2 (Affine equivalence) *Let $(\hat{T}, \hat{\mathcal{V}}, \hat{\mathcal{L}})$ and $(T, \mathcal{V}, \mathcal{L})$ be finite elements and $F_T : \hat{T} \to T$ be a non-degenerate affine map. The finite elements are* affine equivalent *if $\mathcal{F}^*(\mathcal{V}) = \hat{\mathcal{V}}$ and $\mathcal{F}_*(\hat{\mathcal{L}}) = \mathcal{L}$.*

One consequence of affine equivalence is that only a single nodal basis needs to be constructed, and then it can be mapped to each cell in a mesh. Moreover, this idea of equivalence can be extended to some vector-valued elements when certain kinds of Piola mappings are used. In this case, the affine map is the same, but the pull-back and push-forward are appropriately modified. It is also worth stating that not all finite elements generate affine equivalent or Piola-equivalent families. The Lagrange elements are affine equivalent in H^1, but the Hermite and Argyris elements are not. The Raviart–Thomas elements are Piola-equivalent in $H(\text{div})$, while the Mardal–Tai–Winther elements are not.

A dictionary of the finite elements discussed in this chapter is presented in Table 3.1.

3.2 Notation

- The space of polynomials of degree up to and including q on a domain $T \subset \mathbb{R}^d$ is denoted by $\mathcal{P}_q(T)$ and the corresponding d-vector fields by $[\mathcal{P}_q(T)]^d$.
- A finite element space E is called V-conforming if $E \subseteq V$. If not, it is called (V-) nonconforming.
- The elements of $\mathcal{L}$ are usually referred to as the *degrees of freedom* of the element $(T, \mathcal{V}, \mathcal{L})$. When describing finite element families, it is usual to illustrate the degrees of freedom with a certain schematic notation. We summarize the notation used here in the list below and in Figure 3.1.

  Point evaluation. A black sphere (disc) at a point x denotes point evaluation of the function v at that point:

  $$\ell(v) = v(x). \tag{3.2}$$

  For a vector valued function v with d components, a black sphere denotes evaluation of all components and thus corresponds to d degrees of freedom.

Finite element	Short name	Sobolev space	Conforming
(Quintic) Argyris	ARG	H^2	Yes
Arnold–Winther	AW	$H(\mathrm{div};\mathbb{S})$	Yes
Brezzi–Douglas–Marini	BDM	$H(\mathrm{div})$	Yes
Crouzeix–Raviart	CR	H^1	No
Discontinuous Lagrange	DG	L^2	Yes
(Cubic) Hermite	HER	H^2	No
Lagrange	CG	H^1	Yes
Mardal–Tai–Winther	MTW	$H^1/H(\mathrm{div})$	No/Yes
(Quadratic) Morley	MOR	H^2	No
Nédélec first kind	NED^1	$H(\mathrm{curl})$	Yes
Nédélec second kind	NED^2	$H(\mathrm{curl})$	Yes
Raviart–Thomas	RT	$H(\mathrm{div})$	Yes

Table 3.1: A dictionary of the finite elements discussed in this chapter, including full name and the respective (highest order) Sobolev space to which the elements are conforming/nonconforming.

Evaluation of all first derivatives. A dark gray, slightly larger sphere (disc) at a point x denotes point evaluation of all first derivatives of the function v at that point:

$$\ell_i(v) = \frac{\partial v(x)}{\partial x_i}, \quad i = 1, \ldots, d, \tag{3.3}$$

thus corresponding to d degrees of freedom.

Evaluation of all second derivatives. A light gray, even larger sphere (disc) at a point x denotes point evaluation of all second derivatives of the function v at that point:

$$\ell_{ij}(v) = \frac{\partial^2 v(x)}{\partial x_i \partial x_j}, \quad 1 \leqslant i \leqslant j \leqslant d, \tag{3.4}$$

thus corresponding to $d(d+1)/2$ degrees of freedom.

Evaluation of directional component. An arrow at a point x in a direction n denotes evaluation of the vector-valued function v in the direction n at the point x:

$$\ell(v) = v(x) \cdot n. \tag{3.5}$$

The direction n is typically the normal direction of a facet, or a tangent direction of a facet or edge. We will sometimes use an arrow at a point to denote a moment (integration against a weight function) of a component of the function over a facet or edge.

Evaluation of directional derivative. A black line at a point x in a direction n denotes evaluation of the directional derivative of the scalar function v in the direction n at the point x:

$$\ell(v) = \nabla v(x) \cdot n. \tag{3.6}$$

Evaluation of interior moments. A set of concentric spheres (discs) denotes interior moment degrees of freedom; that is, degrees of freedom defined by integration against a weight function over the interior of the domain T. The spheres are colored white-black-white etc.

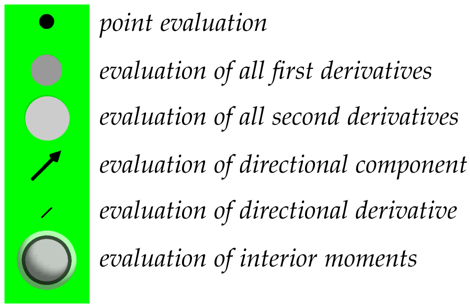


Figure 3.1: Summary of notation used for degrees of freedom. In this example, the three concentric spheres indicate a set of three degrees of freedom defined by interior moments.

We note that, for some of the finite elements presented below, the literature will use different notation and numbering schemes, so that our presentation may be quite different from the original presentation of the elements. In particular, the families of Raviart–Thomas and Nédélec spaces of the first kind are traditionally numbered from 0, while we have followed the more recent scheme from the finite element exterior calculus of numbering from 1.

3.3 H^1 finite elements

The space H^1 is fundamental in the analysis and discretization of weak forms for second-order elliptic problems, and finite element subspaces of H^1 give rise to some of the best-known finite elements. Typically, these elements use C^0 approximating spaces, since a piecewise smooth function on a bounded domain is H^1 if and only if it is continuous (Braess, 2007, Theorem 5.2). We consider the classic Lagrange element, as well as a nonconforming example, the Crouzeix–Raviart space. It is worth noting that the Hermite element considered later is technically only an H^1 element, but can be used as a nonconforming element for smoother spaces. Also, smoother elements such as Argyris may be used to discretize H^1, although this is less common in practice.

3.3.1 The Lagrange element

The best-known and most widely used finite element is the $\mathcal{P}_1$ Lagrange element. This lowest-degree triangle is sometimes called the *Courant* triangle, after the seminal paper by Courant (1943) in which variational techniques are used with the $\mathcal{P}_1$ triangle to derive a finite difference method. Sometimes this is viewed as "the" finite element method, but in fact there is a whole family of elements parametrized by polynomial degree that generalize the univariate Lagrange interpolating polynomials to simplices, boxes, and other shapes. The Lagrange elements of higher degree offer higher order approximation properties. Moreover, these can alleviate locking phenomena observed when using linear elements or give improved discrete stability properties; see Taylor and Hood (1973); Scott and Vogelius (1985).

Definition 3.3 (Lagrange element) *The Lagrange element* (CG_q) *is defined for* $q = 1, 2, \ldots$ *by*

$$T \in \{\text{interval}, \text{triangle}, \text{tetrahedron}\}, \tag{3.7}$$

$$\mathcal{V} = \mathcal{P}_q(T), \tag{3.8}$$

$$\ell_i(v) = v(x^i), \quad i = 1, \ldots, n(q), \tag{3.9}$$

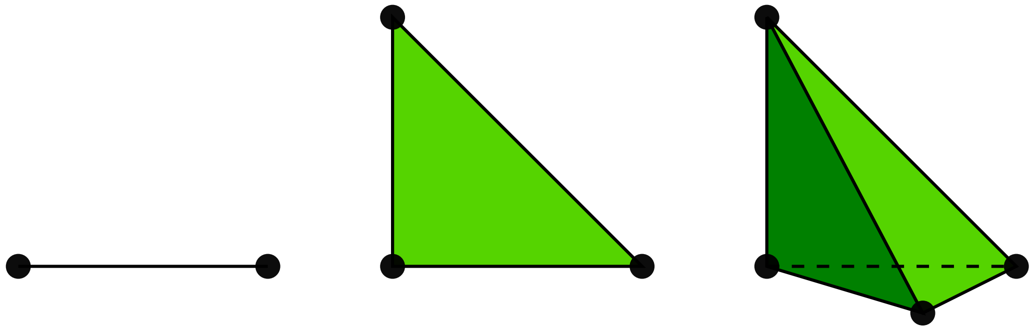


Figure 3.2: The linear Lagrange interval, triangle and tetrahedron.

where $\{x^i\}_{i=1}^{n(q)}$ is an enumeration of points in T defined by

$$x = \begin{cases} i/q, & 0 \leqslant i \leqslant q, & T \text{ interval}, \\ (i/q, j/q), & 0 \leqslant i+j \leqslant q, & T \text{ triangle}, \\ (i/q, j/q, k/q), & 0 \leqslant i+j+k \leqslant q, & T \text{ tetrahedron}. \end{cases} \tag{3.10}$$

The dimension of the Lagrange finite element thus corresponds to the dimension of the complete polynomials of degree q on T and is

$$n(q) = \begin{cases} q+1, & T \text{ interval}, \\ \frac{1}{2}(q+1)(q+2), & T \text{ triangle}, \\ \frac{1}{6}(q+1)(q+2)(q+3), & T \text{ tetrahedron}. \end{cases} \tag{3.11}$$

The definition above presents one choice for the set of points $\{x^i\}$. However, this is not the only possible choice. In general, it suffices that the set of points $\{x^i\}$ is unisolvent and that the boundary points are located so as to allow C^0 assembly. The point set must include the vertices, $q-1$ points on each edge, $\frac{(q-1)(q-2)}{2}$ points per face, and so forth. The boundary points should be placed symmetrically so that the points on adjacent cells match. While numerical conditioning and interpolation properties can be dramatically improved by choosing these points in a clever way (Warburton, 2005), for the purposes of this chapter the points may be assumed to lie on an equispaced lattice; see Figures 3.2, 3.3 and 3.4.

Letting Π_T^q denote the interpolant defined by the above degrees of freedom of the Lagrange element of degree q, we have from Brenner and Scott (2008) that

$$||u - \Pi_T^q u||_{H^1(T)} \leqslant C\, h_T^q |u|_{H^{q+1}(T)}, \quad ||u - \Pi_T^q u||_{L^2(T)} \leqslant C\, h_T^{q+1} |u|_{H^{q+1}(T)}. \tag{3.12}$$

where, here and throughout, C denotes a generic positive constant not depending on h_T but depending on the degree q and the aspect ratio of the simplex, and u is a sufficiently regular function (or vector-field).

Vector-valued or tensor-valued Lagrange elements are usually constructed by using a Lagrange element for each component.

3.3.2 The Crouzeix–Raviart element

The Crouzeix–Raviart element was introduced in Crouzeix and Raviart (1973) as a technique for solving the stationary Stokes equations. The global element space consists of piecewise linear

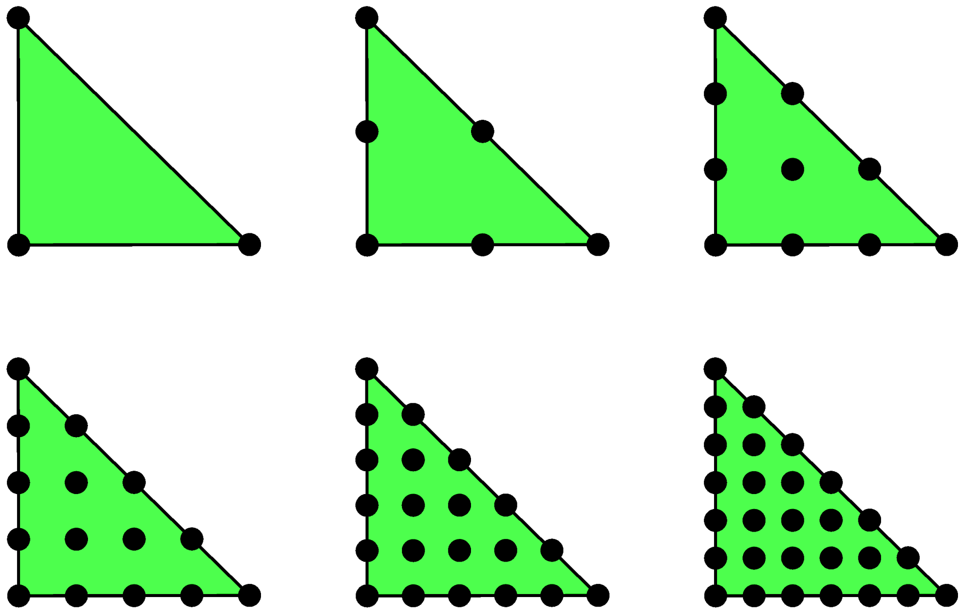


Figure 3.3: The Lagrange CG_q triangle for $q = 1, 2, 3, 4, 5, 6$.

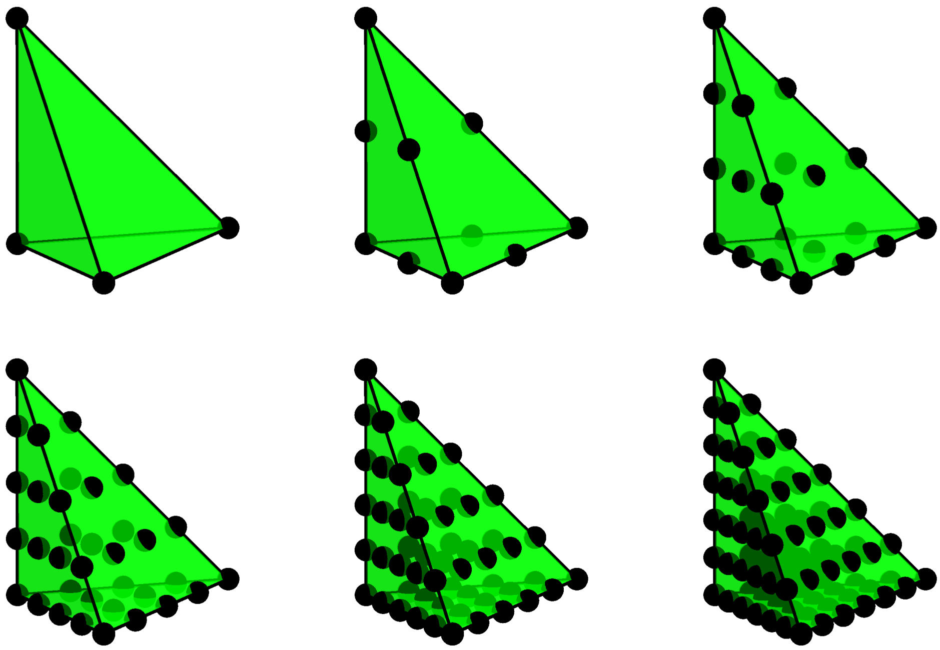


Figure 3.4: The Lagrange CG_q tetrahedron for $q = 1, 2, 3, 4, 5, 6$.

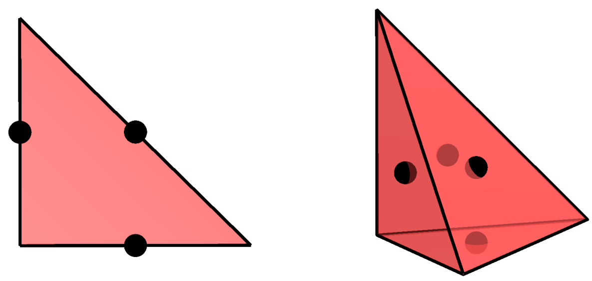


Figure 3.5: Illustration of the Crouzeix–Raviart elements on triangles and tetrahedra. The degrees of freedom are point evaluation at the midpoint of each facet.

polynomials, as for the linear Lagrange element. However, in contrast to the Lagrange element, the global basis functions are not required to be continuous at all points; continuity is only imposed at the midpoint of facets. The element is hence not H^1-conforming, but it is typically used for nonconforming approximations of H^1 functions (and vector fields). Other applications of the Crouzeix–Raviart element includes linear elasticity (Hansbo and Larson, 2003) and Reissner–Mindlin plates (Arnold and Falk, 1989).

Definition 3.4 (Crouzeix–Raviart element) *The (linear) Crouzeix–Raviart element (*CR*) is defined by*

$$T \in \{\text{triangle}, \text{tetrahedron}\}, \tag{3.13}$$

$$\mathcal{V} = \mathcal{P}_1(T), \tag{3.14}$$

$$\ell_i(v) = v(x^i), \quad i = 1, \ldots, n. \tag{3.15}$$

where $\{x^i\}$ are the barycenters (midpoints) of each facet of the domain T.

The dimension of the Crouzeix–Raviart element on $T \subset \mathbb{R}^d$ is thus

$$n = d + 1 \tag{3.16}$$

for $d = 2, 3$.

Letting Π_T denote the interpolation operator defined by the degrees of freedom, the Crouzeix–Raviart element interpolates as the linear Lagrange element (Braess, 2007, Chapter 3.I):

$$||u - \Pi_T u||_{H^1(T)} \leqslant C\, h_T |u|_{H^2(T)}, \quad ||u - \Pi_T u||_{L^2(T)} \leqslant C\, h_T^2 |u|_{H^2(T)}. \tag{3.17}$$

Vector-valued Crouzeix–Raviart elements can be defined by using a Crouzeix–Raviart element for each component, or by using facet normal and facet tangential components at the midpoints of each facet as degrees of freedom. The Crouzeix–Raviart element can be extended to higher odd degrees ($q = 3, 5, 7 \ldots$) (Crouzeix and Falk, 1989).

3.4 $H(\mathrm{div})$ *finite elements*

The Sobolev space $H(\mathrm{div})$ consists of vector fields for which the components and the weak divergence are square-integrable. This is a weaker requirement than for a d-vector field to be in $[H^1]^d$ (for $d \geqslant 2$). This space naturally occurs in connection with mixed formulations of second-order elliptic problems, porous media flow, and elasticity equations. For a finite element family to be $H(\mathrm{div})$-conforming, each component need not be continuous, but the normal component must be continuous. In order

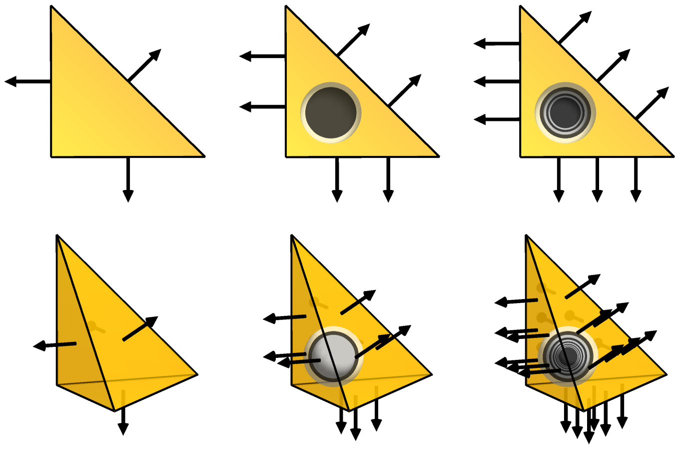


Figure 3.6: Illustration of the degrees of freedom for the first, second and third degree Raviart–Thomas elements on triangles and tetrahedra. The degrees of freedom are moments of the normal component against $\mathcal{P}_{q-1}$ on facets (edges and faces, respectively) and, for the higher degree elements, interior moments against $[\mathcal{P}_{q-2}]^d$. Alternatively, as indicated in this illustration, the moments of normal components may be replaced by point evaluation of normal components.

to ensure such continuity, the degrees of freedom of $H(\mathrm{div})$-conforming elements usually include normal components on element facets.

The two main families of $H(\mathrm{div})$-conforming elements are the Raviart–Thomas and Brezzi–Douglas–Marini elements. These two families are described below. In addition, the Arnold–Winther element discretizing the space of symmetric tensor fields with square-integrable row-wise divergence and the Mardal–Tai–Winther element are included.

3.4.1 *The Raviart–Thomas element*

The Raviart–Thomas element was introduced by Raviart and Thomas (1977). It was the first element to discretize the mixed form of second-order elliptic equations on triangles. Its element space $\mathcal{V}$ is designed so that it is the smallest polynomial space $\mathcal{V} \subset \mathcal{P}_q(T)$, for $q = 1, 2, \ldots$, from which the divergence maps onto $\mathcal{P}_{q-1}(T)$. Shortly thereafter, it was extended to tetrahedra and boxes by Nédélec (1980). It is therefore sometimes referred to as the Raviart–Thomas–Nédélec element. Here, we label both the two- and three-dimensional versions as the Raviart–Thomas element.

The definition given below is based on the one presented by Nédélec (1980) (and Brezzi and Fortin (1991)). The original Raviart–Thomas paper used a slightly different form. Moreover, Raviart and Thomas originally started counting at $q = 0$. Hence, the lowest degree element is traditionally called the RT_0 element. For the sake of consistency, such that a finite element of polynomial degree q is included in $\mathcal{P}_q(T)$, we here label the lowest degree elements by $q = 1$ instead (as did also Nédélec).

Definition 3.5 (Raviart–Thomas element) *The Raviart–Thomas element* (RT_q) *is defined for* $q = 1, 2, \ldots$ *by*

$$T \in \{\text{triangle}, \text{tetrahedron}\}, \tag{3.18}$$

$$\mathcal{V} = [\mathcal{P}_{q-1}(T)]^d + x\mathcal{P}_{q-1}(T), \tag{3.19}$$

$$\mathcal{L} = \begin{cases} \int_f v \cdot n \, p \, \mathrm{d}s, & \text{for a set of basis functions } p \in \mathcal{P}_{q-1}(f) \text{ for each facet } f, \\ \int_T v \cdot p \, \mathrm{d}x, & \text{for a set of basis functions } p \in [\mathcal{P}_{q-2}(T)]^d \text{ for } q \geqslant 2. \end{cases} \tag{3.20}$$

As an example, the lowest degree Raviart–Thomas space on triangles is a three-dimensional space and consists of vector fields of the form

$$v(x) = \alpha + \beta x, \tag{3.21}$$

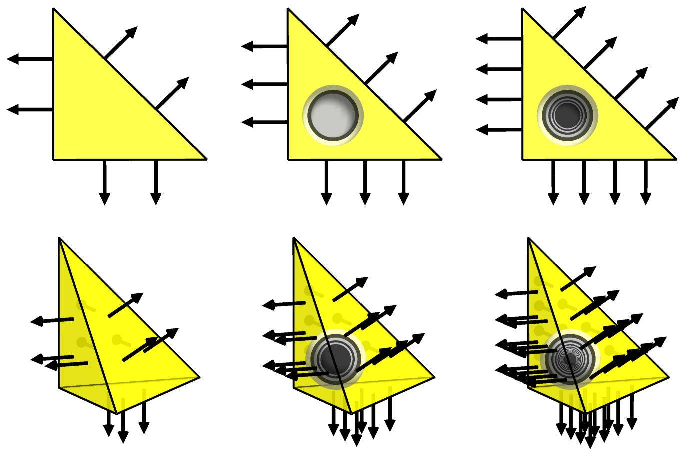


Figure 3.7: Illustration of the first, second and third degree Brezzi–Douglas–Marini elements on triangles and tetrahedra. The degrees of freedom are moments of the normal component against $\mathcal{P}_q$ on facets (edges and faces, respectively) and, for the higher degree elements, interior moments against NED^1_{q-1}. Alternatively, as indicated in this illustration, the moments of normal components may be replaced by point evaluation of normal components.

where α is a vector-valued constant, and β is a scalar constant.

The dimension of RT_q is

$$n(q) = \begin{cases} q(q+2), & T \text{ triangle}, \\ \frac{1}{2}q(q+1)(q+3), & T \text{ tetrahedron}. \end{cases} \tag{3.22}$$

Letting Π_T^q denote the interpolation operator defined by the degrees of freedom above for $q = 1, 2, \ldots$, we have that (Brezzi and Fortin, 1991, Chapter III.3)

$$||u - \Pi_T^q u||_{H(\mathrm{div})(T)} \leqslant C\, h_T^q |u|_{H^{q+1}(T)}, \quad ||u - \Pi_T^q u||_{L^2(T)} \leqslant C\, h_T^q |u|_{H^q(T)}. \tag{3.23}$$

3.4.2 *The Brezzi–Douglas–Marini element*

The Brezzi–Douglas–Marini element was introduced by Brezzi, Douglas and Marini in two dimensions (for triangles) in Brezzi et al. (1985a). The element can be viewed as an alternative to the Raviart–Thomas element using a complete polynomial space. It was later extended to three dimensions (tetrahedra, prisms and cubes) in Nédélec (1986) and Brezzi et al. (1987a). The definition given here is based on that of Nédélec (1986).

The Brezzi–Douglas–Marini element was introduced for mixed formulations of second-order elliptic equations. However, it is also useful for weakly symmetric discretizations of the elastic stress tensor; see Farhloul and Fortin (1997); Arnold et al. (2007).

Definition 3.6 (Brezzi–Douglas–Marini element) *The Brezzi–Douglas–Marini element* (BDM_q) *is defined for* $q = 1, 2, \ldots$ *by*

$$T \in \{\text{triangle}, \text{tetrahedron}\}, \tag{3.24}$$

$$\mathcal{V} = [\mathcal{P}_q(T)]^d, \tag{3.25}$$

$$\mathcal{L} = \begin{cases} \int_f v \cdot n p \, \mathrm{d}s, & \text{for a set of basis functions } p \in \mathcal{P}_q(f) \text{ for each facet } f, \\ \int_T v \cdot p \, \mathrm{d}x, & \text{for a set of basis functions } p \in \mathrm{NED}^1_{q-1}(T) \text{ for } q \geqslant 2. \end{cases} \tag{3.26}$$

where NED^1 *refers to the Nédélec* $H(\mathrm{curl})$ *elements of the first kind, defined below in Section 3.5.1.*

The dimension of BDM_q is

$$n(q) = \begin{cases} (q+1)(q+2), & T \text{ triangle}, \\ \frac{1}{2}(q+1)(q+2)(q+3), & T \text{ tetrahedron}. \end{cases} \tag{3.27}$$

Letting Π_T^q denote the interpolation operator defined by the degrees of freedom for $q = 1, 2, \dots$, we have that (Brezzi and Fortin, 1991, Chapter III.3)

$$||u - \Pi_T^q u||_{H(\mathrm{div})(T)} \leqslant C\, h_T^q |u|_{H^{q+1}(T)}, \quad ||u - \Pi_T^q u||_{L^2(T)} \leqslant C\, h_T^{q+1} |u|_{H^{q+1}(T)}. \tag{3.28}$$

A slight modification of the Brezzi–Douglas–Marini element constrains the element space $\mathcal{V}$ by only allowing normal components on the boundary of polynomial degree $q-1$ (rather than the full polynomial degree q). Such an element was suggested on rectangles by Brezzi et al. (1987b), and the triangular analogue was given in Brezzi and Fortin (1991). In similar spirit, elements with differing degrees on the boundary suitable for varying the polynomial degree between triangles were derived in Brezzi et al. (1985b).

3.4.3 *The Mardal-Tai-Winther element*

The Mardal–Tai–Winther element was introduced in Mardal et al. (2002) as a finite element suitable for the velocity space for both Darcy and Stokes flow in two dimensions. In the Darcy flow equations, the velocity space only requires $H(\mathrm{div})$-regularity. Moreover, discretizations based on H^1-conforming finite elements are typically not stable. On the other hand, for the Stokes equations, the velocity space does stipulate H^1-regularity. The Mardal–Tai–Winther element is $H(\mathrm{div})$-conforming, but H^1-nonconforming. The element was extended to three dimensions in Tai and Winther (2006), but we only present the two-dimensional case here.

Definition 3.7 (Mardal–Tai–Winther element) *The Mardal–Tai–Winther element (*MTW*) is defined by*

$$T = \text{triangle}, \tag{3.29}$$

$$\mathcal{V} = \{v \in [\mathcal{P}_3(T)]^2, \text{ such that } \operatorname{div} v \in \mathcal{P}_0(T) \text{ and } v \cdot n|_f \in \mathcal{P}_1(T) \text{ for each facet } f\}, \tag{3.30}$$

$$\mathcal{L} = \begin{cases} \int_f v \cdot n\, p \,\mathrm{d}s, & \text{for a set of basis functions } p \in \mathcal{P}_1(f) \text{ for each facet } f, \\ \int_f v \cdot t \,\mathrm{d}s, & \text{for each facet } f. \end{cases} \tag{3.31}$$

The dimension of MTW is

$$n = 9. \tag{3.32}$$

Letting Π_T denote the interpolation operator defined by the degrees of freedom, we have that

$$||u - \Pi_T u||_{H^1(T)} \leqslant C\, h_T |u|_{H^2(T)},\ ||u - \Pi_T u||_{H(\mathrm{div})(T)} \leqslant C\, h_T |u|_{H^2(T)},\ ||u - \Pi_T u||_{L^2(T)} \leqslant C\, h_T^2 |u|_{H^2(T)}. \tag{3.33}$$

3.4.4 *The Arnold–Winther element*

The Arnold–Winther element was introduced by Arnold and Winther (2002). This paper presented the first stable mixed (non-composite) finite element for the stress–displacement formulation of linear elasticity. The finite element used for the stress space is what is presented as the Arnold–Winther element here. This finite element is a symmetric tensor element that is row-wise $H(\mathrm{div})$-conforming. The finite element was introduced for a hierarchy of polynomial degrees and extended

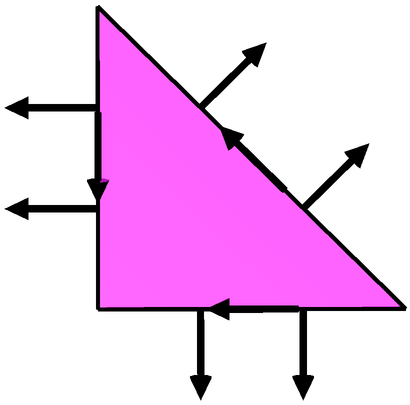


Figure 3.8: Illustration of the Mardal–Tai–Winther element. The degrees of freedom are two moments of the normal component on each facet and one moment of the tangential component on each facet. In this figure, the moments of normal components are illustrated by point evaluation of normal components.

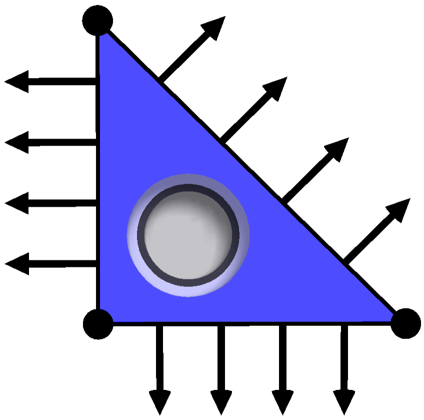


Figure 3.9: Illustration of the Arnold–Winther element. The 24 degrees of freedom are point evaluation at the vertices, the two first moments of the normal component of each row of the tensor field on each facet, and three interior moments.

to three-dimensions in Adams and Cockburn (2005) and Arnold et al. (2008), but we only present the lowest degree two-dimensional case here.

Definition 3.8 (Arnold–Winther element) *The (lowest degree) Arnold–Winther element* (AW) *is defined by*

$$T = \text{triangle}, \tag{3.34}$$

$$\mathcal{V} = \{v \in \mathcal{P}_3(T;\mathbb{S}) : \operatorname{div} v \in \mathcal{P}_1(T;\mathbb{R}^2)\}, \tag{3.35}$$

$$\mathcal{L} = \begin{cases} v(x^k)_{ij}, & \textit{for } 1 \leqslant i \leqslant j \leqslant 2 \textit{ at each vertex } x^k \\ \int_f \sum_{j=1}^2 v_{ij} n_j \, p \, \mathrm{d}s, & \textit{for a set of basis functions } p \in \mathcal{P}_1(f), \textit{ on each facet } f, \ 1 \leqslant i \leqslant 2, \\ \int_T v_{ij} \, \mathrm{d}x, & \textit{for } 1 \leqslant i \leqslant j \leqslant 2. \end{cases} \tag{3.36}$$

The dimension of AW is

$$n = 24. \tag{3.37}$$

Letting Π_T denote the interpolation operator defined by the degrees of freedom, we have that

$$||u - \Pi_T u||_{H(\mathrm{div})(T)} \leqslant C\, h_T^2 |u|_{H^3(T)}, \quad ||u - \Pi_T u||_{L^2(T)} \leqslant C\, h_T^3 |u|_{H^3(T)}. \tag{3.38}$$

3.5 $H(\mathrm{curl})$ *finite elements*

The Sobolev space $H(\mathrm{curl})$ arises frequently in problems associated with electromagnetism. The Nédélec elements, also colloquially referred to as *edge elements*, are much used for such problems, and stand as a premier example of the power of "nonstandard" (meaning not lowest-degree Lagrange) finite elements (Nédélec, 1980, 1986). For a piecewise polynomial to be $H(\mathrm{curl})$-conforming, the tangential component must be continuous. Therefore, the degrees of freedom for $H(\mathrm{curl})$-conforming finite elements typically include tangential components.

There are four families of finite element spaces due to Nédélec, introduced in the papers Nédélec (1980) and Nédélec (1986). The first (1980) paper introduced two families of finite element spaces on tetrahedra, cubes and prisms: one $H(\mathrm{div})$-conforming family and one $H(\mathrm{curl})$-conforming family. These families are known as Nédélec $H(\mathrm{div})$ elements of the *first kind* and Nédélec $H(\mathrm{curl})$ elements of the *first kind*, respectively. The $H(\mathrm{div})$ elements can be viewed as the three-dimensional extension of the Raviart–Thomas elements. (These are therefore presented as Raviart–Thomas elements above.) The first kind Nédélec $H(\mathrm{curl})$ elements are presented below.

The second (1986) paper introduced two more families of finite element spaces: again, one $H(\mathrm{div})$-conforming family and one $H(\mathrm{curl})$-conforming family. These families are known as Nédélec $H(\mathrm{div})$ elements of the *second kind* and Nédélec $H(\mathrm{curl})$ elements of the *second kind*, respectively. The $H(\mathrm{div})$ elements can be viewed as the three-dimensional extension of the Brezzi–Douglas–Marini elements. (These are therefore presented as Brezzi–Douglas–Marini elements above.) The second kind Nédélec $H(\mathrm{curl})$ elements are presented below.

In his two classic papers, Nédélec considered only the three-dimensional case. However, one can also define a two-dimensional curl, and two-dimensional $H(\mathrm{curl})$-conforming finite element spaces. We present such elements as Nédélec elements on triangles here. Although attributing these elements to Nédélec may be dubious, we include them for the sake of completeness.

In many ways, Nédélec's work anticipates the recently introduced finite element exterior calculus presented in Arnold et al. (2006a), where the first kind spaces appear as $\mathcal{P}_q^- \Lambda^k$ spaces and the second kind as $\mathcal{P}_q \Lambda^k$. Moreover, the use of a differential operator (the elastic strain) in Nédélec (1980) to characterize the function space foreshadows the use of differential complexes in Arnold et al. (2006b).

3.5.1 *The Nédélec* $H(\mathrm{curl})$ *element of the first kind*

Definition 3.9 (Nédélec $H(\mathrm{curl})$ element of the first kind) *For $q = 1, 2, \ldots$, define the space*

$$S_q(T) = \{s \in [\mathcal{P}_q(T)]^d \;:\; s(x) \cdot x = 0 \quad \forall\, x \in T\}. \tag{3.39}$$

The Nédélec element of the first kind (NED^1_q) *is defined for $q = 1, 2, \ldots$ in two dimensions by*

$$T = \text{triangle}, \tag{3.40}$$

$$\mathcal{V} = [\mathcal{P}_{q-1}(T)]^2 + S_q(T), \tag{3.41}$$

$$\mathcal{L} = \begin{cases} \int_e v \cdot t\, p \,\mathrm{d}s, & \textit{for a set of basis functions } p \in \mathcal{P}_{q-1}(e) \textit{ for each edge } e, \\ \int_T v \cdot p \,\mathrm{d}x, & \textit{for a set of basis functions } p \in [\mathcal{P}_{q-2}(T)]^2, \textit{ for } q \geqslant 2, \end{cases} \tag{3.42}$$

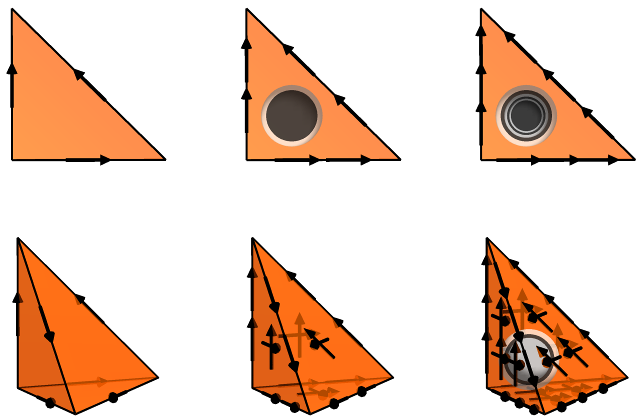


Figure 3.10: Illustration of first, second and third degree Nédélec $H(\mathrm{curl})$ elements of the first kind on triangles and tetrahedra. Note that these elements may be viewed as *rotated* Raviart–Thomas elements. For the first degree Nédélec elements, the degrees of freedom are the average value over edges or, alternatively, the value of the tangential component at the midpoint of edges. Hence the term "edge elements".

where t is the edge tangent; and in three dimensions by

$$T = \text{tetrahedron}, \tag{3.43}$$

$$\mathcal{V} = [\mathcal{P}_{q-1}(T)]^3 + S_q(T), \tag{3.44}$$

$$\mathcal{L} = \begin{cases} \int_e v \cdot t\, p \,\mathrm{d}l, & \text{for a set of basis functions } p \in \mathcal{P}_{q-1}(e) \text{ for each edge } e \\ \int_f v \times n \cdot p \,\mathrm{d}s, & \text{for a set of basis functions } p \in [\mathcal{P}_{q-2}(f)]^2 \text{ for each face } f, \text{ for } q \geqslant 2, \\ \int_T v \cdot p \,\mathrm{d}x, & \text{for a set of basis functions } p \in [\mathcal{P}_{q-3}]^3, \text{ for } q \geqslant 3. \end{cases} \tag{3.45}$$

The dimension of NED^1_q is

$$n(q) = \begin{cases} q(q+2), & T \text{ triangle}, \\ \frac{1}{2}q(q+2)(q+3), & T \text{ tetrahedron}. \end{cases} \tag{3.46}$$

Letting Π^q_T denote the interpolation operator defined by the degrees of freedom above, we have that (Nédélec, 1980, Theorem 2)

$$||u - \Pi^q_T u||_{H(\mathrm{curl})(T)} \leqslant C\, h^q_T |u|_{H^{q+1}(T)}, \quad ||u - \Pi^q_T u||_{L^2(T)} \leqslant C\, h^q_T |u|_{H^q(T)}. \tag{3.47}$$

3.5.2 *The $H(\mathrm{curl})$ Nédélec element of the second kind*

Definition 3.10 (Nédélec $H(\mathrm{curl})$ element of the second kind) *The Nédélec element of the second kind (NED^2_q) is defined for $q = 1, 2, \ldots$ in two dimensions by*

$$T = \text{triangle}, \tag{3.48}$$

$$\mathcal{V} = [\mathcal{P}_q(T)]^2, \tag{3.49}$$

$$\mathcal{L} = \begin{cases} \int_e v \cdot t\, p \,\mathrm{d}s, & \text{for a set of basis functions } p \in \mathcal{P}_q(e) \text{ for each edge } e, \\ \int_T v \cdot p \,\mathrm{d}x, & \text{for a set of basis functions } p \in \mathrm{RT}_{q-1}(T), \text{ for } q \geqslant 2. \end{cases} \tag{3.50}$$

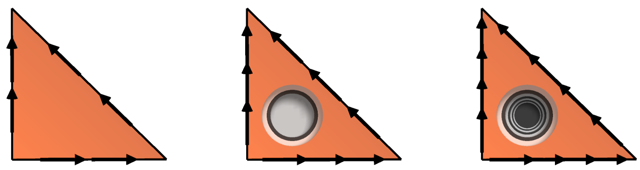


Figure 3.11: Illustration of first, second and third degree Nédélec $H(\text{curl})$ elements of the second kind on triangles. Note that these elements may be viewed as *rotated* Brezzi–Douglas–Marini elements.

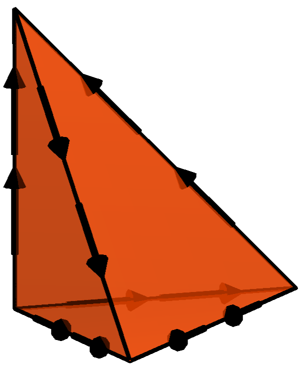


Figure 3.12: Illustration of the first degree Nédélec $H(\text{curl})$ elements of the second kind on tetrahedra.

where t is the edge tangent, and in three dimensions by

$$T = \text{tetrahedron}, \tag{3.51}$$

$$\mathcal{V} = [\mathcal{P}_q(T)]^3, \tag{3.52}$$

$$\mathcal{L} = \begin{cases} \int_e v \cdot t\, p \,\mathrm{d}l, & \textit{for a set of basis functions } p \in \mathcal{P}_q(e) \textit{ for each edge } e, \\ \int_f v \cdot p \,\mathrm{d}s, & \textit{for a set of basis functions } p \in \mathrm{RT}_{q-1}(f) \textit{ for each face } f, \textit{ for } q \geqslant 2 \\ \int_T v \cdot p \,\mathrm{d}x, & \textit{for a set of basis functions } p \in \mathrm{RT}_{q-2}(T), \textit{ for } q \geqslant 3. \end{cases} \tag{3.53}$$

The dimension of $\mathrm{NED}^2{}_q$ is

$$n(q) = \begin{cases} (q+1)(q+2), & T \text{ triangle}, \\ \frac{1}{2}(q+1)(q+2)(q+3), & T \text{ tetrahedron}. \end{cases} \tag{3.54}$$

Letting Π_T^q denote the interpolation operator defined by the degrees of freedom above, we have that (Nédélec, 1986, Proposition 3)

$$||u - \Pi_T^q u||_{H(\mathrm{curl})(T)} \leqslant C\, h_T^q |u|_{H^{q+1}(T)}, \quad ||u - \Pi_T^q u||_{L^2(T)} \leqslant C\, h_T^{q+1} |u|_{H^{q+1}(T)}. \tag{3.55}$$

3.6 L^2 finite elements

By L^2 elements, one usually refers to finite element spaces where the elements are not in C^0. Such elements naturally occur in mixed formulations of the Poisson equation, Stokes flow, and elasticity. Alternatively, such elements can be used for nonconforming methods imposing the desired continuity weakly instead of directly. The discontinuous Galerkin (DG) methods provide a typical example. In this case, the numerical flux of element facets is assembled as part of the weak form; numerous variants of DG methods have been defined with different numerical fluxes. DG methods were originally developed for hyperbolic problems but have been successfully applied to many elliptic and parabolic problems. Moreover, the decoupling of each individual element provides an increased opportunity for parallelism and *hp*-adaptivity.

3.6.1 Discontinuous Lagrange

Definition 3.11 (Discontinuous Lagrange element) *The discontinuous Lagrange element* (DG_q) *is defined for* $q = 0, 1, 2, \ldots$ *by*

$$T \in \{\text{interval}, \text{triangle}, \text{tetrahedron}\}, \tag{3.56}$$

$$\mathcal{V} = \mathcal{P}_q(T), \tag{3.57}$$

$$\ell_i(v) = v(x^i), \tag{3.58}$$

where $\{x^i\}_{i=1}^{n(q)}$ *is an enumeration of points in* T *defined by*

$$x = \begin{cases} i/q, & 0 \leqslant i \leqslant q, & T \text{ interval}, \\ (i/q, j/q) & 0 \leqslant i + j \leqslant q, & T \text{ triangle}, \\ (i/q, j/q, k/q) & 0 \leqslant i + j + k \leqslant q, & T \text{ tetrahedron}. \end{cases} \tag{3.59}$$

The dimension of DG_q is

$$n(q) = \begin{cases} q + 1, & T \text{ interval}, \\ \frac{1}{2}(q+1)(q+2), & T \text{ triangle}, \\ \frac{1}{6}(q+1)(q+2)(q+3), & T \text{ tetrahedron}. \end{cases} \tag{3.60}$$

Letting Π_T^q denote the interpolation operator defined by the degrees of freedom above, the interpolation properties of the DG_q elements of degree q are:

$$||u - \Pi_T^q u||_{L^2(T)} \leqslant C\, h_T^{q+1} |u|_{H^{q+1}(T)}. \tag{3.61}$$

3.7 H^2 finite elements

The H^2 elements are commonly used in the approximation of fourth-order problems, or for other spaces requiring at least C^1 continuity. Due to the restrictive nature of the continuity requirement, conforming elements are often of a high polynomial degree, but lower degree nonconforming elements have proven to be successful. Therefore, we here consider the conforming Argyris element and the nonconforming Hermite and Morley elements.

3.7.1 The Argyris element

The Argyris element (Argyris et al., 1968; Ciarlet, 2002) is based on the space $\mathcal{P}_5(T)$ of quintic polynomials over some triangle T. It can be pieced together with full C^1 continuity between elements and C^2 continuity at the vertices of a triangulation.

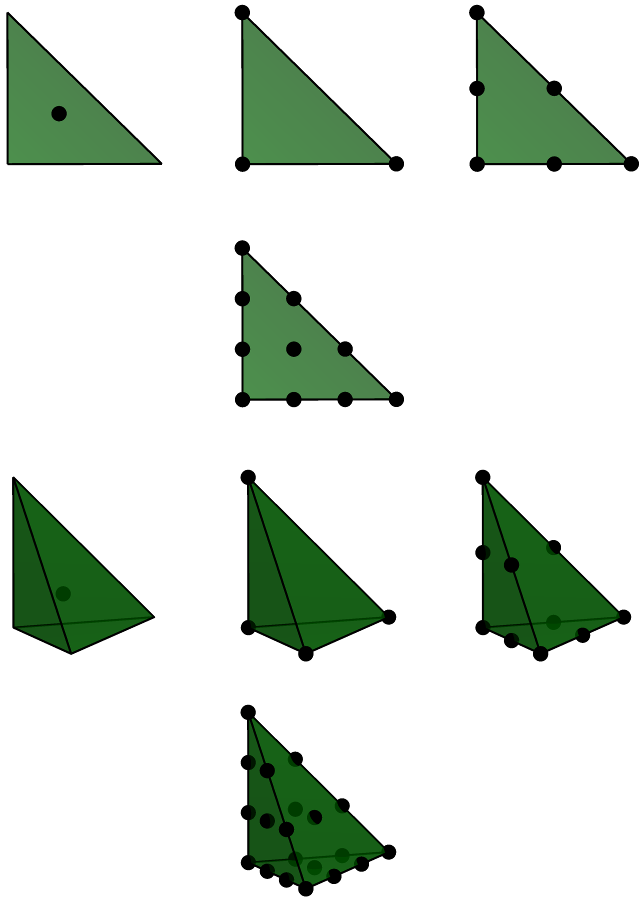


Figure 3.13: Illustration of the zeroth, first, second and third degree discontinuous Lagrange elements on triangles and tetrahedra. The degrees of freedom may be chosen arbitrarily as long as they span the dual space $\mathcal{V}'$. Here, the degrees of freedom have been chosen to be identical to those of the standard Lagrange finite element, with the difference that the degrees of freedom are viewed as *internal* to the element.

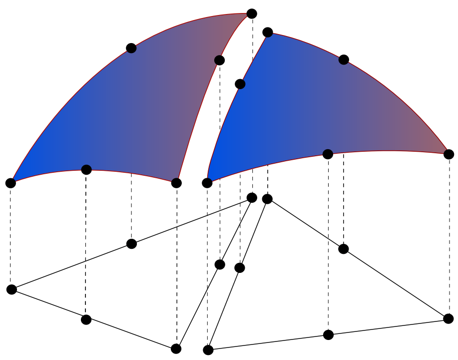


Figure 3.14: All degrees of freedom of a discontinuous Lagrange finite element are internal to the element, which means that no global continuity is imposed by these elements. This is illustrated here for discontinuous quadratic Lagrange elements.

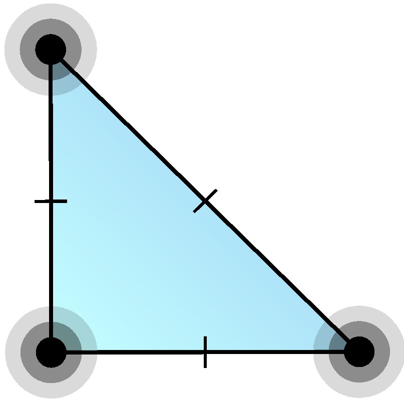


Figure 3.15: The quintic Argyris triangle. The degrees of freedom are point evaluation, point evaluation of both first derivatives and point evaluation of all three second derivatives at the vertices of the triangle, and evaluation of the normal derivative at the midpoint of each edge.

Definition 3.12 (Argyris element) *The (quintic) Argyris element* (ARG_5) *is defined by*

$$T = \text{triangle}, \tag{3.62}$$

$$\mathcal{V} = \mathcal{P}_5(T), \tag{3.63}$$

$$\mathcal{L} = \begin{cases} v(x^i), & \text{for each vertex } x^i, \\ \operatorname{grad} v(x^i)_j, & \text{for each vertex } x^i, \text{ and each component } j, \\ D^2 v(x^i)_{jk}, & \text{for each vertex } x^i, \text{ and each component } jk, j \leqslant k, \\ \operatorname{grad} v(m^i) \cdot n, & \text{for each edge midpoint } m^i. \end{cases} \tag{3.64}$$

The dimension of ARG_5 is

$$n = 21. \tag{3.65}$$

Letting Π_T denote the interpolation operator defined by the degrees of freedom above, the interpolation properties of the (quintic) Argyris elements are (Braess, 2007, Chapter II.6):

$$||u - \Pi_T u||_{H^2(T)} \leqslant C\, h_T^4 |u|_{H^6(T)}, \quad ||u - \Pi_T u||_{H^1(T)} \leqslant C\, h_T^5 |u|_{H^6(T)}, \quad ||u - \Pi_T u||_{L^2(T)} \leqslant C\, h_T^6 |u|_{H^6(T)}. \tag{3.66}$$

The normal derivatives in the dual basis for the Argyris element prevent it from being affine-interpolation equivalent. This prevents the nodal basis from being constructed on a reference cell and affinely mapped. Recent work by Domínguez and Sayas (2008) develops a transformation that corrects this issue and requires less computational effort than directly forming the basis on each cell in a mesh. The Argyris element can be generalized to polynomial degrees higher than quintic, still giving C^1 continuity with C^2 continuity at the vertices (Šolín et al., 2004).

3.7.2 *The Hermite element*

The Hermite element generalizes the classic cubic Hermite interpolating polynomials on the line segment (Ciarlet, 2002). Hermite-type elements appear in the finite element literature almost from the beginning, appearing at least as early as the classic paper by Ciarlet and Raviart (1972). They have long been known as useful C^1-nonconforming elements (Braess, 2007; Ciarlet, 2002). Under affine mappings, the Hermite elements form *affine-interpolation equivalent* families (Brenner and Scott, 2008).

On the triangle, the space of cubic polynomials is ten-dimensional, and the ten degrees of freedom for the Hermite element are point evaluation at the triangle vertices and barycenter, together with the

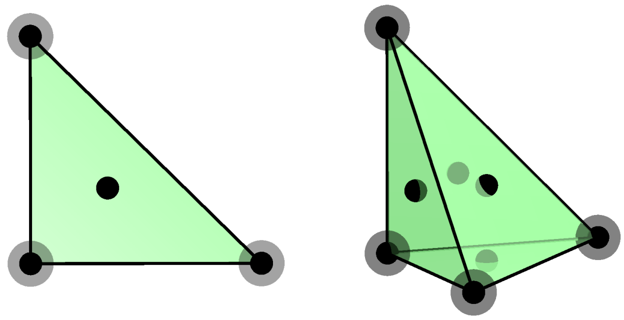


Figure 3.16: The cubic Hermite triangle and tetrahedron. The degrees of freedom are point evaluation at the vertices and the barycenter, and evaluation of both first derivatives at the vertices.

components of the gradient evaluated at the vertices. The generalization to tetrahedra is analogous.

Definition 3.13 (Hermite element) *The (cubic) Hermite element (*HER*) is defined by*

$$T \in \{\text{interval}, \text{triangle}, \text{tetrahedron}\}, \tag{3.67}$$

$$\mathcal{V} = \mathcal{P}_3(T), \tag{3.68}$$

$$\mathcal{L} = \begin{cases} v(x^i), & \textit{for each vertex } x^i, \\ \operatorname{grad} v(x^i)_j, & \textit{for each vertex } x^i, \textit{ and each component } j, \\ v(b), & \textit{for the barycenter } b \textit{ (of the faces in 3D).} \end{cases} \tag{3.69}$$

The dimension of HER is

$$n = \begin{cases} 10, & T \text{ triangle}, \\ 20, & T \text{ tetrahedron}. \end{cases} \tag{3.70}$$

Letting Π_T denote the interpolation operator defined by the degrees of freedom above, the interpolation properties of the (cubic) Hermite elements are:

$$||u - \Pi_T u||_{H^1(T)} \leqslant C\, h_T^3 |u|_{H^4(T)}, \quad ||u - \Pi_T u||_{L^2(T)} \leqslant C\, h_T^4 |u|_{H^4(T)}. \tag{3.71}$$

Unlike the cubic Hermite functions on a line segment, the cubic Hermite triangle and tetrahedron cannot be patched together in a fully C^1 fashion. The cubic Hermite element can be extended to higher degree (Brenner and Scott, 2008).

3.7.3 The Morley element

The Morley triangle defined in Morley (1968) is a simple H^2-nonconforming quadratic element that is used in fourth-degree problems. The function space $\mathcal{V}$ is simply $\mathcal{P}_2(T)$, the six-dimensional space of quadratics. The degrees of freedom consist of pointwise evaluation at each vertex and the normal derivative at each edge midpoint. It is interesting to note that the Morley triangle is neither C^1 nor even C^0, yet it is suitable for fourth-order problems, and is the simplest known element for this purpose.

The Morley element was first introduced to the engineering literature by Morley (1968, 1971). In the mathematical literature, Lascaux and Lesaint (1975) considered it in the context of the patch test in a study of plate-bending elements. Recent applications of the Morley element include Huang et al. (2008); Ming and Xu (2006).

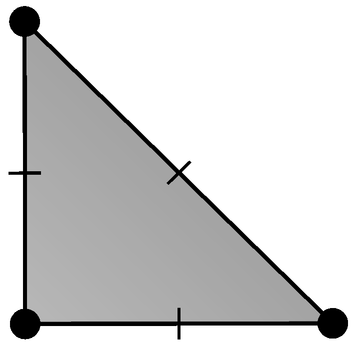


Figure 3.17: The quadratic Morley triangle. The degrees of freedom are point evaluation at the vertices and evaluation of the normal derivative at the midpoint on each edge.

Definition 3.14 (Morley element) *The (quadratic) Morley element (*MOR*) is defined by*

$$T = \text{triangle}, \tag{3.72}$$
$$\mathcal{V} = \mathcal{P}_2(T), \tag{3.73}$$
$$\mathcal{L} = \begin{cases} v(x^i), & \textit{for each vertex } x^i, \\ \operatorname{grad} v(m^i) \cdot n, & \textit{for each edge midpoint } m^i. \end{cases} \tag{3.74}$$

The dimension of the Morley element is

$$n = 6. \tag{3.75}$$

Letting Π_T denote the interpolation operator defined by the degrees of freedom above, the interpolation properties of the (quadratic) Morley elements are:

$$||u - \Pi_T u||_{H^1(T)} \leqslant C\, h_T^2 |u|_{H^3(T)}, \quad ||u - \Pi_T u||_{L^2(T)} \leqslant C\, h_T^3 |u|_{H^3(T)}. \tag{3.76}$$

3.8 *Enriching finite elements*

If U, V are linear spaces, one can define a new linear space W by

$$W = \{w = u + v \, : \, u \in U, v \in V\}. \tag{3.77}$$

Here, we choose to call such a space W an *enriched space*.

The enrichment of a finite element space can lead to improved stability properties, especially for mixed finite element methods. Examples include the enrichment of the Lagrange element with bubble functions for use with the Stokes equations or enriching the Raviart–Thomas element for linear elasticity (Arnold et al., 1984a,b). Bubble functions have since been used for many different applications. We here define a *bubble element* for easy reference. Notable examples of the use of a bubble element include:

The MINI element for the Stokes equations. In the lowest degree case, the linear vector Lagrange element is enriched with the cubic vector bubble element for the velocity approximation (Arnold et al., 1984b).

The PEERS element for weakly symmetric linear elasticity. Each row of the stress tensor is approximated by the lowest degree Raviart–Thomas element enriched by the curl of the cubic bubble element (Arnold et al., 1984a).

Definition 3.15 (Bubble element) *The bubble element (B_q) is defined for $q \geqslant (d+1)$ by*

$$T \in \{\text{interval}, \text{triangle}, \text{tetrahedron}\}, \tag{3.78}$$

$$\mathcal{V} = \{v \in \mathcal{P}_q(T) : v|_{\partial T} = 0\}, \tag{3.79}$$

$$\ell_i(v) = v(x^i), \quad i = 1, \ldots, n(q). \tag{3.80}$$

where $\{x^i\}_{i=1}^{n(q)}$ is an enumeration of the points[3] *in T defined by*

$$x = \begin{cases} (i+1)/q, & 0 \leqslant i \leqslant q-2, & T \text{ interval}, \\ ((i+1)/q, (j+1)/q), & 0 \leqslant i+j \leqslant q-3, & T \text{ triangle}, \\ ((i+1)/q, (j+1)/q, (k+1)/q), & 0 \leqslant i+j+k \leqslant q-4, & T \text{ tetrahedron}. \end{cases} \tag{3.81}$$

The dimension of the Bubble element is

$$n(q) = \begin{cases} q-1, & T \text{ interval}, \\ \frac{1}{2}(q-2)(q-1), & T \text{ triangle}, \\ \frac{1}{6}(q-3)(q-2)(q-1), & T \text{ tetrahedron}. \end{cases} \tag{3.82}$$

3.9 Finite element exterior calculus

It has recently been demonstrated that many of the finite elements that have been discovered or invented over the years can be formulated and analyzed in a common unifying framework as special cases of a more general class of finite elements. This new framework is known as *finite element exterior calculus* and is summarized in Arnold et al. (2006a). In finite element exterior calculus, two finite element spaces $\mathcal{P}_q\Lambda^k(T)$ and $\mathcal{P}_q^-\Lambda^k(T)$ are defined for general simplices T of dimension $d \geqslant 1$. The element $\mathcal{P}_q\Lambda^k(T)$ is the space of polynomial differential k-forms[4] on T with degrees of freedom chosen to ensure continuity of the trace on facets. When these elements are interpreted as regular elements, by a suitable identification between differential k-forms and scalar- or vector-valued functions, one obtains a series of well-known elements for $0 \leqslant k \leqslant d \leqslant 3$. In Table 3.2, we summarize the relation between these elements and the elements presented above in this chapter[5].

3.10 Summary

In the table below, we summarize the list of elements discussed in this chapter. For brevity, we include element degrees only up to and including $q = 3$. For higher degree elements, we refer to the script `dolfin-plot` available as part of FEniCS, which can be used to easily plot the degrees of freedom for a wide range of elements:

Bash code

[3] Any other basis for the dual space of $\mathcal{V}$ will work just as well.

[4] A differential k-form ω on a domain Ω maps each point $x \in \Omega$ to an alternating k-form ω_x on the tangent space $T_x(\Omega)$ of Ω at the point x. One can show that for $d = 3$, the differential k-forms correspond to scalar-, vector-, vector-, and scalar-valued functions for $k = 0, 1, 2, 3$ respectively. Thus, we may identify for example both $\mathcal{P}_q\Lambda^1$ and $\mathcal{P}_q\Lambda^2$ on a tetrahedron with the vector-valued polynomials of degree at most q on the tetrahedron.

[5] The finite elements $\mathcal{P}_q\Lambda^k(T)$ and $\mathcal{P}_q^-\Lambda^k(T)$ have been implemented for general values of k, q and $d = 1, 2, 3, 4, \ldots$ as part of the FEniCS *Exterior* package available from `http://launchpad.net/exterior`.

$\mathcal{P}_q\Lambda^k$

k	$d=1$	$d=2$	$d=3$
0	CG_q	CG_q	CG_q
1	DG_q	$\mathrm{NED}_q^{2,\mathrm{curl}}$	$\mathrm{NED}_q^{2,\mathrm{curl}}$
2	—	DG_q	BDM_q
3	—	—	DG_q

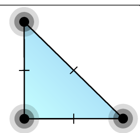
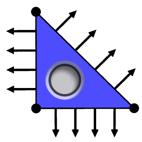
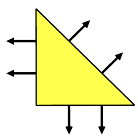
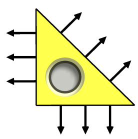
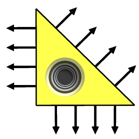
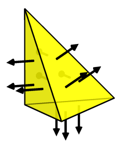
$\mathcal{P}_q^-\Lambda^k$

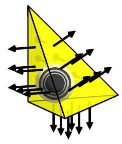
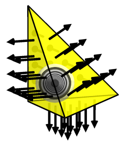
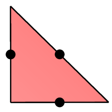
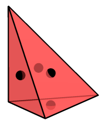
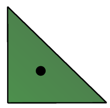
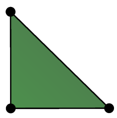
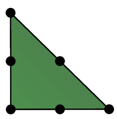
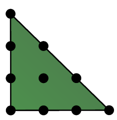
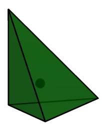
k	$d=1$	$d=2$	$d=3$
0	CG_q	CG_q	CG_q
1	DG_{q-1}	$\mathrm{NED}_q^{1,\mathrm{curl}}$	$\mathrm{NED}_q^{1,\mathrm{curl}}$
2	—	DG_{q-1}	RT_q
3	—	—	DG_{q-1}

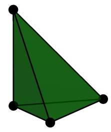
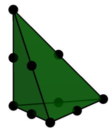
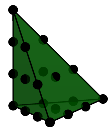
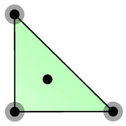
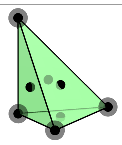
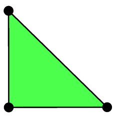
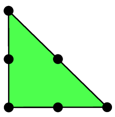
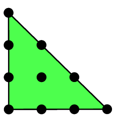
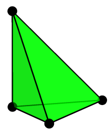
Table 3.2: Relationships between the finite elements $\mathcal{P}_q\Lambda^k$ and $\mathcal{P}_q^-\Lambda^k$ defined by finite element exterior calculus and their more traditional counterparts using the numbering and labeling of this chapter.

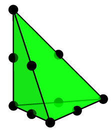
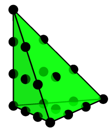
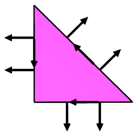
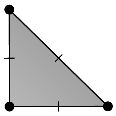
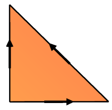
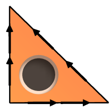
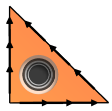
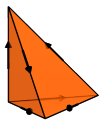
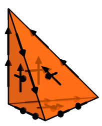
```
$ dolfin-plot BDM tetrahedron 3
$ dolfin-plot N1curl triangle 4
$ dolfin-plot CG tetrahedron 5
```

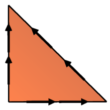
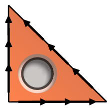
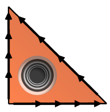
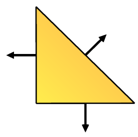
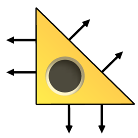
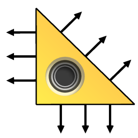
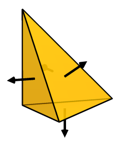
Elements indicated with at $(*)$ in the table below are fully supported by FEniCS.

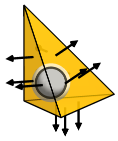
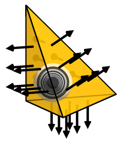
Element family	Notation	Illustration	Dimension	Description
(Quintic) Argyris	ARG_5 (2D)		$n=21$	$\mathcal{P}_5$ (scalar); 3 point values, 3×2 derivatives, 3×3 second derivatives, 3 directional derivatives
Arnold–Winther	AW (2D)		$n=24$	$\mathcal{P}_3(T;\mathbb{S})$ (matrix) with linear divergence; 3×3 point values, 12 normal components, 3 interior moments
Brezzi–Douglas–Marini $(*)$	BDM_1 (2D)		$n=6$	$[\mathcal{P}_1]^2$ (vector); 6 normal components
Brezzi–Douglas–Marini $(*)$	BDM_2 (2D)		$n=12$	$[\mathcal{P}_2]^2$ (vector); 9 normal components, 3 interior moments
Brezzi–Douglas–Marini $(*)$	BDM_3 (2D)		$n=20$	$[\mathcal{P}_3]^2$ (vector); 12 normal components, 8 interior moments
Brezzi–Douglas–Marini $(*)$	BDM_1 (3D)		$n=12$	$[\mathcal{P}_1]^3$ (vector); 12 normal components

Brezzi–Douglas–Marini (∗)	BDM_2 (3D)		$n = 30$	$[\mathcal{P}_2]^3$ (vector); 24 normal components, 6 interior moments
Brezzi–Douglas–Marini (∗)	BDM_3 (3D)		$n = 60$	$[\mathcal{P}_3]^3$ (vector); 40 normal components, 20 interior moments
Crouzeix–Raviart (∗)	CR_1 (2D)		$n = 3$	$\mathcal{P}_1$ (scalar); 3 point values
Crouzeix–Raviart (∗)	CR_1 (3D)		$n = 4$	$\mathcal{P}_1$ (scalar); 4 point values
Discontinuous Lagrange (∗)	DG_0 (2D)		$n = 1$	$\mathcal{P}_0$ (scalar); 1 point value
Discontinuous Lagrange (∗)	DG_1 (2D)		$n = 3$	$\mathcal{P}_1$ (scalar); 3 point values
Discontinuous Lagrange (∗)	DG_2 (2D)		$n = 6$	$\mathcal{P}_2$ (scalar); 6 point values
Discontinuous Lagrange (∗)	DG_3 (2D)		$n = 10$	$\mathcal{P}_3$ (scalar); 10 point values
Discontinuous Lagrange (∗)	DG_0 (3D)		$n = 1$	$\mathcal{P}_0$ (scalar); 1 point value

Discontinuous Lagrange $(*)$	DG_1 (3D)		$n = 4$	$\mathcal{P}_1$ (scalar); 4 point values
Discontinuous Lagrange $(*)$	DG_2 (3D)		$n = 10$	$\mathcal{P}_2$ (scalar); 10 point values
Discontinuous Lagrange $(*)$	DG_3 (3D)		$n = 20$	$\mathcal{P}_3$ (scalar); 20 point values
(Cubic) Hermite	HER (2D)		$n = 10$	$\mathcal{P}_3$ (scalar); 4 point values, 3×2 derivatives
(Cubic) Hermite	HER (3D)		$n = 20$	$\mathcal{P}_3$ (scalar); 8 point values, 4×3 derivatives
Lagrange $(*)$	CG_1 (2D)		$n = 3$	$\mathcal{P}_1$ (scalar); 3 point values
Lagrange $(*)$	CG_2 (2D)		$n = 6$	$\mathcal{P}_2$ (scalar); 6 point values
Lagrange $(*)$	CG_3 (2D)		$n = 10$	$\mathcal{P}_3$ (scalar); 10 point values
Lagrange $(*)$	CG_1 (3D)		$n = 4$	$\mathcal{P}_1$ (scalar); 4 point values

Lagrange $(*)$	CG_2 (3D)		$n = 10$	$\mathcal{P}_2$ (scalar); 10 point values
Lagrange $(*)$	CG_3 (3D)		$n = 20$	$\mathcal{P}_2$ (scalar); 20 point values
Mardal–Tai–Winther	MTW (2D)		$n = 9$	$[\mathcal{P}_2]^2$ (vector); with constant divergence and linear normal components; 6 moments of normal components, 3 moments of tangential components
(Quadratic) Morley	MOR (2D)		$n = 6$	$\mathcal{P}_2$ (scalar); 3 point values, 3 directional derivatives
Nédélec 1st kind $H(\text{curl})$ $(*)$	NED^1_1 (2D)		$n = 3$	$[\mathcal{P}_0]^2 + S_1$ (vector); 3 tangential components
Nédélec 1st kind $H(\text{curl})$ $(*)$	NED^1_2 (2D)		$n = 8$	$[\mathcal{P}_1]^2 + S_2$ (vector); 6 tangential components, 2 interior moments
Nédélec 1st kind $H(\text{curl})$ $(*)$	NED^1_3 (2D)		$n = 15$	$[\mathcal{P}_2]^2 + S_3$ (vector); 9 tangential components, 6 interior moments
Nédélec 1st kind $H(\text{curl})$ $(*)$	NED^1_1 (3D)		$n = 6$	$[\mathcal{P}_0]^3 + S_1$ (vector); 6 tangential components
Nédélec 1st kind $H(\text{curl})$ $(*)$	NED^1_2 (3D)		$n = 20$	$[\mathcal{P}_1]^3 + S_2$ (vector); 20 tangential components

Nédélec 1st kind $H(\text{curl})$ $(*)$	NED_3^1 (3D)		$n = 45$	$[\mathcal{P}_2]^3 + \mathcal{S}_3$ (vector); 42 tangential components, 3 interior moments
Nédélec 2nd kind $H(\text{curl})$ $(*)$	NED_1^2 (2D)		$n = 6$	$[\mathcal{P}_1]^2$ (vector); 6 tangential components
Nédélec 2nd kind $H(\text{curl})$ $(*)$	NED_2^2 (2D)		$n = 12$	$[\mathcal{P}_2]^2$ (vector); 9 tangential components, 3 interior moments
Nédélec 2nd kind $H(\text{curl})$ $(*)$	NED_3^2 (2D)		$n = 20$	$[\mathcal{P}_3]^2$ (vector); 12 tangential components, 8 interior moments
Nédélec 2nd kind $H(\text{curl})$ $(*)$	NED_1^2 (3D)		$n = 12$	$[\mathcal{P}_1]^3$ (vector); 12 tangential components
Raviart–Thomas $(*)$	RT_1 (2D)		$n = 3$	$[\mathcal{P}_0]^2 + x\mathcal{P}_0$ (vector); 3 normal components
Raviart–Thomas $(*)$	RT_2 (2D)		$n = 8$	$[\mathcal{P}_1]^2 + x\mathcal{P}_1$ (vector); 6 normal components, 2 interior moments
Raviart–Thomas $(*)$	RT_3 (2D)		$n = 15$	$[\mathcal{P}_2]^2 + x\mathcal{P}_2$ (vector); 9 normal components, 6 interior moments
Raviart–Thomas $(*)$	RT_1 (3D)		$n = 4$	$[\mathcal{P}_0]^3 + x\mathcal{P}_0$ (vector); 4 normal components

Raviart–Thomas (∗)	RT_2 (3D)		$n = 15$	$[\mathcal{P}_1]^3 + x\mathcal{P}_1$ (vector); 12 normal components, 3 interior moments
Raviart–Thomas (∗)	RT_3 (3D)		$n = 36$	$[\mathcal{P}_2]^3 + x\mathcal{P}_2$ (vector); 24 normal components, 12 interior moments