A Reference Solution for the Lorenz System on [0, 1000]

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Abstract. In this note, we demonstrate that the Lorenz system is computable on intervals of arbitrary length using high order and high precision numerics. We present a reference solution for the Lorenz system on the interval [0, 1000] computed with a numerical method of order 200 and 420 digit precision.

Keywords: Lorenz system, computability, error analysis, reference solution
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INTRODUCTION

In a now classic paper [1], Edward Lorenz studied a system of three ordinary differential equations:

$$\begin{cases} \dot{x} &= \sigma(y-x), \\ \dot{y} &= rx - y - xz, \\ \dot{z} &= xy - bz, \end{cases}$$

where $\sigma = 10$, $b = 8/3$, and $r = 28$. He observed that computed solutions of these systems are very sensitive to small perturbations in the initial data. Computed solutions are also very sensitive to numerical errors and the size of the time step used to compute solutions. It has also been observed that numerical solutions may fail to converge when the size of the time step is decreased. For this reason, it is commonly believed that the Lorenz system is not computable, or computable only over short time intervals. However, as demonstrated in [2], where accurate solutions are computed on the interval [0, 30], and in [3], where accurate solutions are computed on the interval [0, 48], the Lorenz system is computable over intervals of moderate length. In this note, we explain that solutions are computable on intervals of arbitrary length, if a high order and high precision numerical method is used.

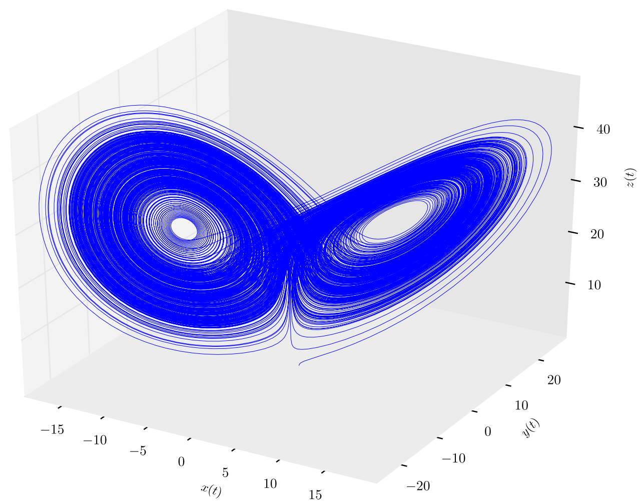

FIGURE 1. Phase portrait of the solution of the Lorenz system on the time interval $[0, 1000]$ for $u(0) = (1, 0, 0)$.

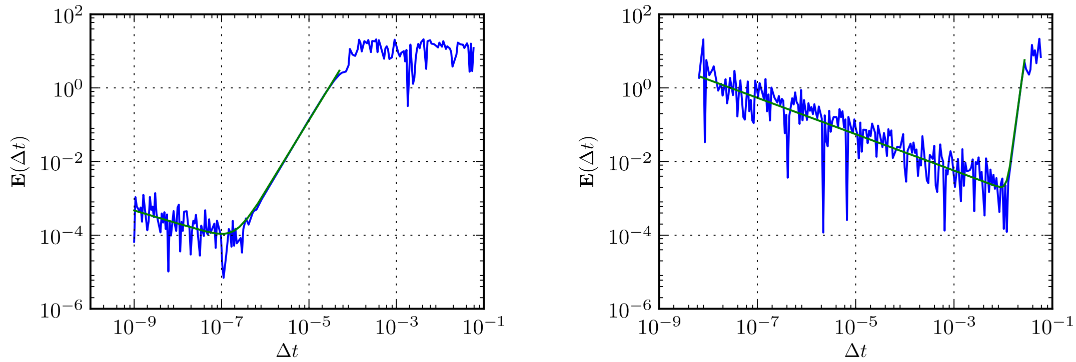

FIGURE 2. Error at time $T = 30$ for the cG(1) solution (left) and at time $T = 40$ for the cG(5) solution (right) of the Lorenz system. The slopes of the green lines are $-0.35 \approx -1/2$ and $1.95 \approx 2$ for the cG(1) method. For the cG(5) method, the slopes are $-0.49 \approx -1/2$ and $10.00 \approx 10$.

COMPUTABILITY OF THE LORENZ SYSTEM

In [4], we study the computability of the Lorenz system in detail. We find that the error in a numerical solution is the sum of three contributions:

$$\mathbf{E} = \mathbf{E}_D + \mathbf{E}_G + \mathbf{E}_C \sim S_D(T)\|U(0) - u(0)\| + S_G(T)\Delta t^p + S_C(T)\Delta t^{-1/2}\varepsilon,$$

where Δt is the size of the time step, p is the order of convergence of the numerical method, ε is the machine precision, and $S_D(T)$, $S_G(T)$, $S_C(T)$ are stability factors that measure the influence of errors in initial data, errors due to the discretization scheme, and computational (round-off) errors respectively.

Note that according to this estimate, the error contribution related to the numerical method decreases as Δt decreases as expected. However, round-off errors increase when the time step length is decreased. This explains why it is non-trivial to compute accurate solutions of the Lorenz system; by decreasing the time step, discretization errors are reduced, but at the same time round-off errors will increase and may dominate discretization errors. Indeed, for fixed numerical precision, the error will initially decrease with decreasing time step, and then start to increase as shown in Figure 2.

In Figure 3, we plot the computational stability factor $S_C(T)$ as function of the final time T. At time $T = 50$, we see that the stability factor is of size $\sim 10^{16}$, which limits the computability of the Lorenz system with standard double precision to $T \approx 50$. If instead one uses 420 digits of precision, the computability is limited to $T \approx 1050$, see [4]. By estimating the growth rate of the stability factors over longer time intervals (by solving an associated dual problem), one may conclude that the computability (the endtime to which one can compute accurate solutions) of the Lorenz system is limited to

$$T \approx 2.5\, n_\varepsilon,$$

where n_ε is the number of significant decimal digits. Increasing the numerical precision results in a (polynomial) increase of the computational time. Thus, the computability of the Lorenz system is limited only by available computing resources in terms of CPU time and memory.

REFERENCE SOLUTION ON [0, 1000]

Using fixed floating-point precision, $\varepsilon = 10^{-420}$, fixed time step[1] length, $\Delta t = 0.0037$, and the cG(100) finite element method [5, 6] (a method of order 200), we compute the solution of the Lorenz system with $T = 1000$. The solution is plotted in Figure 4 and reference values are given in Table 1.

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[1] One may show that the optimal size of the time step is of the order $\varepsilon^{1/(p+1/2)}$.

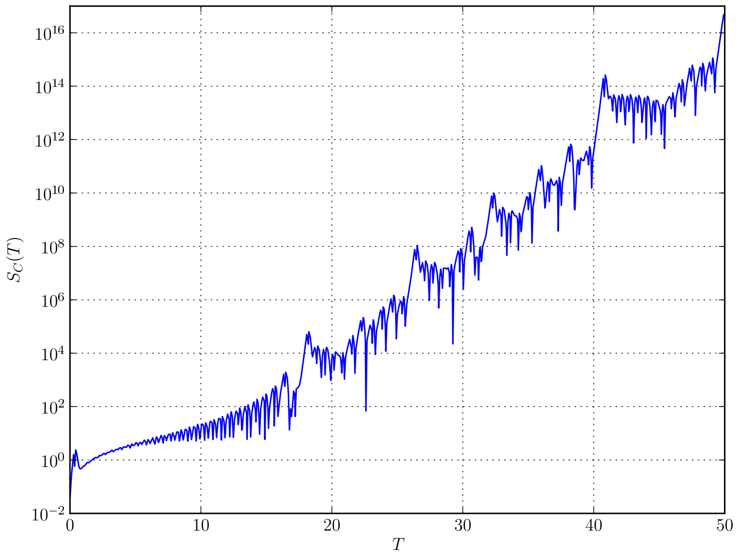

FIGURE 3. Growth of the stability factor S_C accounting for accumulation of round-off errors on the time interval $[0,50]$.

TABLE 1. Sample values of the reference solution.

t	x	y	z
0	1	0	0
1	−9.408450567056036	−9.096199071186479	28.581627624392340
2	−7.876082549991658	−8.761621817314939	24.990260995565926
3	−8.143999245434069	−6.942058961947722	28.120426584688346
4	−9.453542010189011	−10.430214212303325	26.938025375071280
5	−6.974570472684818	−7.021060890822531	25.119616492127594
10	−5.857685382424090	−5.831082486426101	23.932132987027561
20	−8.021143613317436	−11.905464749171751	19.856374858398414
30	−3.892637337379485	0.274019816217374	27.866107798922574
40	−0.194731422418737	0.181102990970540	17.234465691306429
50	16.246275544569393	23.171100649775568	30.162707065608277
100	12.286006677018985	16.581464262811753	26.597525090160012
200	−0.428878362621397	−0.812388566743750	8.141562045191055
300	−7.946536271496051	−1.159949771727215	33.524744262421997
400	5.943245586080340	9.672638471715088	16.076376870383008
500	−11.118772890892362	−15.011144913050707	25.223573344110680
600	13.982585780713302	12.606312490063734	35.686835498440018
700	−7.974669839553495	−2.477557738987132	32.428427138140691
800	0.882587574884531	1.111923175877187	15.724886069017078
900	15.450276601479965	21.458347336788307	29.834655787376352
1000	12.537229584422063	12.842013118843248	31.953398729274273

To verify the solution, we compute solutions with fixed Δt, fixed ε, and increase the order of the method, $p = 20, 40, \ldots, 180, 198, 200$ to obtain a converging sequence of solutions. The solutions computed with $\mathrm{cG}(99)$ and $\mathrm{cG}(100)$ (of order 198 and 200 respectively) agree on the interval $[0,1025]$. The computations were performed using the finite element package DOLFIN [7] together with the multi-precision library GMP [8].

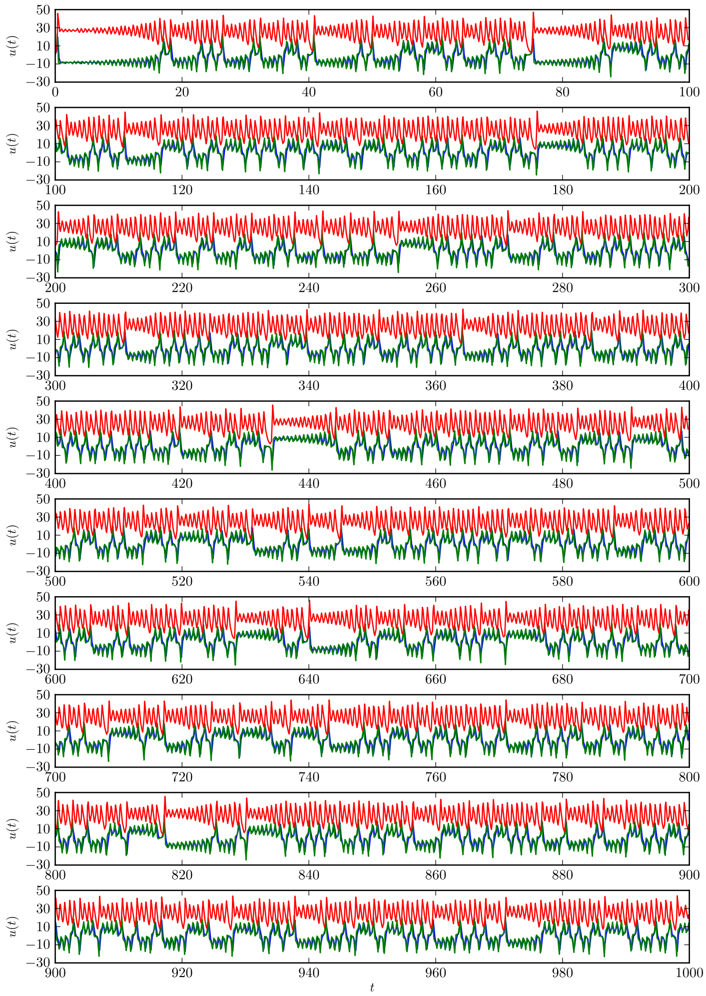

FIGURE 4. The three components of the Lorenz system on the interval $[0, 1000]$ with the x and y components plotted in blue/green respectively (bottom) and the z component in red (top).

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