A Reference Solution for the Lorenz System on \([0, 1000]\)

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Abstract. In this note, we demonstrate that the Lorenz system is computable on intervals of arbitrary length using high order and high precision numerics. We present a reference solution for the Lorenz system on the interval \([0, 1000]\) computed with a numerical method of order 200 and 420 digit precision.

Keywords: Lorenz system, computability, error analysis, reference solution

PACS: 02.60.Cb, 02.60.Jh, 02.70.Dh

INTRODUCTION

In a now classic paper [1], Edward Lorenz studied a system of three ordinary differential equations:

\[
\begin{align*}
\dot{x} &= \sigma (y - x), \\
\dot{y} &= r x - y - xz, \\
\dot{z} &= xy - bz,
\end{align*}
\]

where \(\sigma = 10\), \(b = 8/3\), and \(r = 28\). He observed that computed solutions of these systems are very sensitive to small perturbations in the initial data. Computed solutions are also very sensitive to numerical errors and the size of the time step used to compute solutions. It has also been observed that numerical solutions may fail to converge when the size of the time step is decreased. For this reason, it is commonly believed that the Lorenz system is not computable, or computable only over short time intervals. However, as demonstrated in [2], where accurate solutions are computed on the interval \([0, 30]\), and in [3], where accurate solutions are computed on the interval \([0, 48]\), the Lorenz system is computable over intervals of moderate length. In this note, we explain that solutions are computable on intervals of arbitrary length, if a high order and high precision numerical method is used.

\[\text{FIGURE 1. Phase portrait of the solution of the Lorenz system on the time interval } [0, 1000] \text{ for } u(0) = (1, 0, 0).\]
FIGURE 2. Error at time $T = 30$ for the cG(1) solution (left) and at time $T = 40$ for the cG(5) solution (right) of the Lorenz system. The slopes of the green lines are $-0.35 \approx -1/2$ and $1.95 \approx 2$ for the cG(1) method. For the cG(5) method, the slopes are $-0.49 \approx -1/2$ and $10.00 \approx 10$.

COMPUTABILITY OF THE LORENZ SYSTEM

In [4], we study the computability of the Lorenz system in detail. We find that the error in a numerical solution is the sum of three contributions:

$$E = E_D + E_G + E_C \sim S_D(T)\|U(0) - u(0)\| + S_G(T)\Delta t^p + S_C(T)\Delta t^{-1/2} \epsilon,$$

where $\Delta t$ is the size of the time step, $p$ is the order of convergence of the numerical method, $\epsilon$ is the machine precision, and $S_D(T)$, $S_G(T)$, $S_C(T)$ are stability factors that measure the influence of errors in initial data, errors due to the discretization scheme, and computational (round-off) errors respectively.

Note that according to this estimate, the error contribution related to the numerical method decreases as $\Delta t$ decreases as expected. However, round-off errors increase when the time step length is decreased. This explains why it is non-trivial to compute accurate solutions of the Lorenz system; by decreasing the time step, discretization errors are reduced, but at the same time round-off errors will increase and may dominate discretization errors. Indeed, for fixed numerical precision, the error will initially decrease with decreasing time step, and then start to increase as shown in Figure 2.

In Figure 3, we plot the computational stability factor $S_C(T)$ as function of the final time $T$. At time $T = 50$, we see that the stability factor is of size $\sim 10^{16}$, which limits the computability of the Lorenz system with standard double precision to $T \approx 50$. If instead one uses 420 digits of precision, the computability is limited to $T \approx 1050$, see [4]. By estimating the growth rate of the stability factors over longer time intervals (by solving an associated dual problem), one may conclude that the computability (the endtime to which one can compute accurate solutions) of the Lorenz system is limited to

$$T \approx 2.5 n_\epsilon,$$

where $n_\epsilon$ is the number of significant decimal digits. Increasing the numerical precision results in a (polynomial) increase of the computational time. Thus, the computability of the Lorenz system is limited only by available computing resources in terms of CPU time and memory.

REFERENCE SOLUTION ON [0, 1000]

Using fixed floating-point precision, $\epsilon = 10^{-420}$, fixed time step length, $\Delta t = 0.0037$, and the cG(100) finite element method [5, 6] (a method of order 200), we compute the solution of the Lorenz system with $T = 1000$. The solution is plotted in Figure 4 and reference values are given in Table 1.

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1 One may show that the optimal size of the time step is of the order $\epsilon^{1/(p+1/2)}$. 
To verify the solution, we compute solutions with fixed \( \Delta t \), fixed \( \epsilon \), and increase the order of the method, \( p = 20, 40, \ldots, 180, 198, 200 \) to obtain a converging sequence of solutions. The solutions computed with \( \text{cG}(99) \) and \( \text{cG}(100) \) (of order 198 and 200 respectively) agree on the interval \([0, 1025]\). The computations were performed using the finite element package DOLFIN [7] together with the multi-precision library GMP [8].
ACKNOWLEDGMENTS

This work is supported by an Outstanding Young Investigator grant from the Research Council of Norway, NFR 180450. This work is also supported by a Center of Excellence grant from the Research Council of Norway to the Center for Biomedical Computing at Simula Research Laboratory.

REFERENCES