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Mathematics and Computation

All thought is a kind of computation. (Hobbes)

How much of mathematics is computational in spirit? One may argue that large parts of mathematics, including calculus, real and complex analysis, Fourier analysis, algebra and number theory, have essential computational (symbolic and numerical) aspects. Further, it may appear that classical geometry according to Euclid or topology have essential qualitative aspects, while again analytic geometry and algebraic topology are largely computational. We know that the scientific revolution occurred together with the shift from Euclidean geometry to Cartesian analytic computational geometry, which initiated the birth of calculus by Leibniz and Newton. We also know that from this basis the industrial society has developed with mass-production of material goods. In the information society of today the computer is changing science, technology, and our lives by offering mass-processing of information. When Google responds in a second to a search request, this is the result of a mathematical computation, as is the weather report, the scanner image at the hospital, and countless other applications.

One may thus ask what the impact of the computer and computation is on mathematics and mathematics education today and what it may be tomorrow?
In V. Arnold's article in the Intelligencer, great mathematicians are characterized by their inclination to replace "blind calculations" by "clear ideas". But are these two really exclusive? Isn't it more true that they go hand in hand? That a clear idea can often have a clear quantitative computational expression, and vice versa?

We believe so, and following this belief we have developed a reformed mathematics education from beginning undergraduate to advanced graduate level based on a synthesis of computation and mathematical analysis. We refer to this as the Body&Soul Applied Mathematics Education Reform Project, with *Body* representing computation and *Soul* representing mathematical analysis. Body&Soul includes a series of text books published by Springer, together with educational material and software available on the project homepage www.phi.chalmers.se/bodysoul/.

In our recent book *Dreams of Calculus: Perspectives on Mathematics Education*, we give an introduction to the Body&Soul Project, by presenting some key ideas in a non-technical way in order to hopefully reach a larger audience. We here connect to one of the central themes discussed in Dreams, a theme related to a famous article from 1960 by the physicist E.P. Wigner (Nobel Prize winner in 1963), *The Unreasonable Effectiveness of Mathematics in the Natural Sciences*, which may be viewed as the rationale behind much of the mathematics education today. Wigner followed up on Galileo's idea that *the book of nature is written in the language of mathematics*. Wigner argued as follows: Consider mathematical models such as (i) Newton's equations of motion and gravity, (ii) Schrödinger's equation for quantum mechanics, (iii) Maxwell's equations for electromagnetics, (iv) the Navier–Stokes equations for fluid dynamics, (v) the wave equation for acoustics, (vi) Navier's equations for elasticity and (vii) Einstein's equation for gravity. Each one of these systems of differential equations can be expressed analytically in a couple of lines and yet each one seems to describe a very rich world of phenomena. The equations seem to confirm Leibniz'

idea that we live in the *best of worlds*, which is a world of maximal complexity governed by the simplest possible laws, which couples to Wigner's paradoxical *unreasonable effectiveness*. But as in every paradox, there is a catch; the equations are difficult to solve by analytical mathematics. The solutions may be very complex and thus difficult to describe using analytical mathematical formulas.

In some rare cases, analytical solutions are known. For Newton's equations, the solution of the two-body problem was derived by Newton, which justified Kepler's laws. The success with the two-body problem rocketed Newton to instant fame, and gave mathematics an enormous boost. It appeared that Man using mathematics could take up competition with God and now, with no more limits to human understanding of the world, the industrial revolution could get started. The paradigm of our time is largely the same, with the secret of life being uncovered in genomic biology.

The fact that neither Newton, nor anyone else after him, could tackle even the simple three-body problem analytically, did not seem to take away the enthusiasm. The situation is largely the same concerning the solution of the other differential equations; few analytical solutions are known and, as in the case of Newton's equations, the few known analytical solutions are subject to intensive worship.

Today, the computer combined with computational methods is changing the game completely. Solutions of the equations (i)–(vii) can be computed, with more or less computational work. And, even more importantly, this capability is now becoming available to massive numbers of engineers, scientists and others. Leibniz' *best of worlds* is now being reformulated into a basic question of algorithmic information theory comparing the length of a computer program with the length (or depth) of the information it may produce. Since the mathematical models (i)–(vii) are short, the computer programs for their computational solution can also be made short, while the richness or amount of information of their output can be very large.

For example, solving the Navier–Stokes equations computationally, we obtain turbulent very complex solutions as soon as the fluid viscosity is small. Just as in the real *best of worlds*, where the simple interaction of many fluid particles creates a complex turbulent flow.
But there is a catch also in the computational form of the *best of worlds*; sometimes the computational work required becomes unattainable, even with the combined power of all existing and conceivable computers. This happens if we seek to compute pointwise values in space–time of turbulent Navier–Stokes solutions, for which only certain mean values are computable (and observable). Direct computational solution of Schrödinger's equation for a system of atoms or molecules also quickly becomes problematic for any computer, since the number of space dimensions involved is equal to three times the number of electrons (viewing the nuclei as fixed). With 100 electrons, we thus face a problem in 300 space dimensions, which is a nightmare for discretization and computation. Not even ten electrons is directly feasible. Schrödinger's himself understood that his equation contains too much information, as do pointwise values of a turbulent flow, and we thus have to go for computing certain mean values of, for example, an electron density instead of point values of the wavefunction for each individual electron. A computational method of this form was developed by the chemist W. Kohn (Nobel Prize winner in 1998), which is now routinely used in molecular simulations.

The situation is the same for the other equations in the list (i)–(vii); computational solution remains a main challenge in many cases. To tackle this problem, adaptive computational methods are now being developed. Adaptive methods compute certain output quantities (e.g. mean values in a turbulent flow) with a given accuracy at minimal computational cost.

Using adaptive methods, it is today possible to reliably compute certain aspects of turbulent flow, e.g, the mean value of the drag coefficient of a bluff body such as a car, using a standard laptop computer.
However, the main challenges remain: In simulations of protein folding (a basic process of life) using molecular dynamics, the time scales range from femtoseconds (10_15) to microseconds (10_6), which makes direct simulation impossible. We encounter similar difficulties with large ranges of scales in space–time is present in many applications. Computational solution of Einstein's equation is also essentially an open problem.
Returning to Wigner's *unreasonable effectiveness*, we may rephrase this today as the *reasonable effectiveness of computational mathematics*. Using computational mathematics, it is possible to model and simulate many complex phenomena in many different areas of application, but we have to pay a computational cost (time and money).

Thus, the solution of (i)–(vii) does not come for free, which would have been *unreasonable*. We have to pay a price, which after all is *reasonable*. The development of computational methods over the last 50 years since the birth of the computer has only scratched the surface of the field of computation, and offers a rich field for mathematical exploration with the goal of *getting more for less*, following the basic principle of our modern market economy.

A main challenge today is thus to extend the realm of the *reasonable effectiveness of computational mathematics* to equations such as Schrödinger's and Einstein's, which are now (partly) out of reach. We hope that the mathematical community is ready to take on the challenge.

A final word about mathematics education: Since computation is now opening entirely new possibilities in mathematical modeling of real world phenomena, mathematics education needs to be reformed on all levels to properly take advantage of the new *best world of computational mathematics*.

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For a more detailed account of the authors' thoughts on mathematics education, see their new book:
Dreams of Calculus.
Perspectives on Mathematics Education.