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## Unified framework for finite element assembly

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**Abstract:** At the heart of any finite element simulation is the assembly of matrices and vectors from discrete variational forms. We propose a general interface between problem-specific and general-purpose components of finite element programs. This interface is called Unified Form-assembly Code (UFC). A wide range of finite element problems is covered, including mixed finite elements and discontinuous Galerkin methods. We discuss how the UFC interface enables implementations of variational form evaluation to be independent of mesh and linear algebra components. UFC does not depend on any external libraries, and is released into the public domain.

**Keywords:** finite elements; assembly; implementation; code generation; UFC; unified form-assembly code.

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## 1 Introduction

Software for solving physical problems has traditionally been tailored to the problem at hand, often resulting in computationally very efficient special-purpose codes. However, experience has shown that such codes may be difficult and costly to extend to new problems. To decrease turn-over time from problem definition to its numerical solution, scientific code writers have to an increasingly larger extent tried to create general libraries, containing common numerical algorithms applicable to a wide range of problems. Such libraries can reduce the size of the application code dramatically and hide implementation details. In the field of finite element solution of partial differential equations, many general and successful libraries have emerged during the last couple of decades, e.g., Cactus, Cogito, COMSOL Multiphysics, Deal.II, Diffpack, DOLFIN (FEniCS), Getfem++, Kaskade, Sundance, and UG (see the reference list for papers and websites).

General finite element libraries implement many standard mathematical and numerical concepts, but the software components are often not as carefully designed as their mathematical counterparts. From a software engineering point of view it is important to achieve clear separation of the various software components that build up a finite element library, such that each component can be replaced separately. Not only does this offer greater flexibility for application and library developers, but it also makes the software easier to maintain, especially under changing requirements of several developers in long-term projects. These arguments have received much attention by developers of general finite element libraries in recent years (see, e.g., Bangerth et al., 2007; Bastian et al., 2008a).

Well designed libraries provide clear interfaces to represent this separation. Typically, the application code uses functions or objects in the interface to perform basic ‘high level’ steps of the solution process. Problem-specific details, such as the variational form, the mesh and coefficients are passed through the interface to the library to compute a solution. Such libraries and their interfaces are generally referred to as Problem Solving Environments (PSEs).

However, one fundamental issue in designing such software libraries is how to separate problem-specific

code from general library code. Some components, such as computational meshes and linear algebra, may be implemented as reusable components (e.g., as a set of C++ classes) with well-defined interfaces. However, other components, such as variational forms, are intrinsically problem-specific. As a result, those components must either be implemented and provided by the user or *generated* automatically by the library from a high-level description of the variational form. In either case, it becomes important to settle on a well-defined interface for how the library should communicate with those problem-specific components.

The design of such an interface is the subject of the present paper. We propose a C++ interface called UFC, which provides an interface between general reusable finite element libraries and problem-specific code. In other libraries, the implementation of finite elements and variational forms is usually tied to the specific mesh, matrix and vector format in use, while in UFC we have strived to decouple these concepts. Furthermore, the interface is designed to allow for a variety of elements such as continuous and discontinuous Lagrange, Nedgelec and Raviart-Thomas elements.

To make a successful interface, one needs a sufficiently general framework for the underlying mathematical structures and operations. The software interface in the current paper relies on a more general view of variational forms and finite element assembly than commonly found in textbooks. We therefore precisely state the mathematical background and notation in Sections 2 and 3.

UFC is significantly inspired by our needs in the tools FFC, SFC, and DOLFIN, which are software units within FEniCS, see Kirby and Logg (2006, 2007), Alnæs and Mardal (2008), and Logg (2007). The interplay between these tools and UFC is explained in Section 4, which provides additional and more specific motivation for the design of UFC. Highlights of the interface are covered in Section 5. Section 6 contains some examples of high-level specifications of variational forms with the form compilers FFC and SFC, which automatically generate code compatible with the UFC interface for computing element matrices and vectors. We also explain how the interface can be used with existing libraries.

We here note that as a result of the UFC interface, the two form compilers FFC and SFC may now be used

interchangeably since both generate code conforming to the UFC interface. These form compilers were developed separately and independently. The work on UFC was initially inspired by our efforts to unify the interfaces for these form compilers.

### 1.1 Related work

One major reason for the success of general finite element libraries is that many widely different physical problems can be solved by quite short application codes utilising the same library. The opposite strategy, i.e., one application utilising different alternative libraries, has received less attention. For example, an application might want to use an adaptive mesh data structure and its functionality from one library, a very efficient assembly routine from another library, basic iterative methods from, e.g., PETSc, combined with a preconditioner from Trilinos or Hypre. To make this composition a true plug-and-play operation, the various libraries would need to conform to a unified interface to the basic operations needed in finite element solvers. Alternatively, low level interfaces can be implemented with thin wrapper code to connect separate software components.

In numerical linear algebra, the BLAS and LAPACK interfaces have greatly simplified code writing. By expressing operations in the application code in terms of BLAS and LAPACK calls, and using the associated data (array) formats, one program can be linked to different implementations of the BLAS and LAPACK operations. Despite the great success of this approach, the idea has to little extent been explored in other areas of computational science. One recent example is Easyviz (Ring et al., 2008), a thin unified interface to curve plotting and 2D/3D scalar- and vector-field visualisation. This interface allows an application program to use a MATLAB-compatible syntax to create graphics, independently of the choice of graphics package (Gnuplot, Grace, MATLAB, VTK, VisIt, OpenDX, etc.). Another example is GLAS (Meerbergen, 2008), a community initiative to specify a general interface for linear algebra libraries. GLAS can be viewed as an extension and modernisation of the BLAS/LAPACK idea, utilising powerful constructs offered by C++.

Within finite elements, DUNE (Bastian et al., 2008a, 2008b) is a very promising attempt to define unified interfaces between application code and libraries for finite element computing. DUNE provides interfaces to data structures and solution algorithms, especially finite element meshes and iterative solution methods for linear systems. In principle, one can write an application code independently of the mesh data structure and the matrix solution method. DUNE does not directly address interfaces between the finite element problem definition (element matrices and vectors), and the assembly process, which is the topic of the present paper. Another difference between DUNE and our UFC interface is the choice of programming technology used in the interface:

DUNE relies heavily on inlining via C++ templates for efficient single-point data retrieval, while UFC applies pointers to chunks of data. However, our view of a finite element mesh can easily be adapted to the DUNE-Grid interface. The DUNE-FEM module (under development) represents interfaces to various discretisation operators and serves some of the purposes of the UFC interface, though being technically quite different.

In the finite element world, there are many competing libraries, each with their own specialties. Thin interfaces offering only the least common denominator functionality do not support special features for special problems and may therefore be met with criticism. Thick interfaces, trying to incorporate ‘all’ functionality in ‘all’ libraries, become too complicated to serve the original purpose of simplifying software development. Obtaining community consensus for the thickness and syntax of a unified interface is obviously an extremely challenging process. The authors of this paper suggest another approach: a small group of people defines a thin (and hence efficient and easy-to-use) interface, they make the software publicly available together with a detailed documentation, and demonstrate its advantages. This is our aim with the present paper.

## 2 Finite element discretisation

### 2.1 The finite element

A finite element is mathematically defined as a triplet consisting of a polygon, a polynomial function space, and a set of linear functionals, see Ciarlet (1978). Given that the dimension of the function space and the number of the (linearly independent) linear functionals are equal, the finite element is uniquely defined. Hence, we will refer to a finite element as a collection of

- a polygon  $K$
- a polynomial space  $\mathcal{P}_K$  on  $K$
- a set of linearly independent linear functionals, the *degrees of freedom*,  $\ell_i : \mathcal{P}_K \rightarrow \mathbb{R}$ ,  $i = 1, 2, \dots, n_K$ .

With this definition the basis functions  $\{\phi_i^K\}_{i=1}^{n_K}$  are obtained by solving the following system of equations,

$$\ell_i(\phi_j^K) = \delta_{ij}, \quad i, j = 1, 2, \dots, n_K. \quad (1)$$

The computation of such a nodal basis can be automated, given (a basis for) the polynomial space  $\mathcal{P}_K$  and the set of linear functionals  $\{\ell_i\}_{i=1}^{n_K}$ , see Kirby (2004) and Alnæs and Mardal (2008).

### 2.2 Variational forms

Consider the Poisson problem  $-\nabla \cdot (w \nabla u) = f$  with Dirichlet boundary conditions on a domain  $\Omega \subset \mathbb{R}^d$ . Multiplying by a test function  $v \in V_h$  and integrating by

parts, one obtains the variational problem

$$\int_{\Omega} w \nabla v \cdot \nabla u_h \, dx = \int_{\Omega} v f \, dx, \quad \forall v \in V_h, \quad (2)$$

for the approximation  $u_h \in V_h$ . If  $w, f \in W_h$  for some discrete function space<sup>1</sup>  $W_h$  we may thus write equation (2) as

$$a(v, u_h; w) = L(v; f) \quad \forall v \in V_h, \quad (3)$$

where the trilinear form  $a : V_h \times V_h \times W_h \rightarrow \mathbb{R}$  is given by

$$a(v, u_h; w) = \int_{\Omega} w \nabla v \cdot \nabla u_h \, dx \quad (4)$$

and the bilinear form  $L : V_h \times W_h \rightarrow \mathbb{R}$  is given by

$$L(v; f) = \int_{\Omega} v f \, dx. \quad (5)$$

Note here that  $a$  is *bilinear* for any given fixed  $w \in W_h$  and  $L$  is *linear* for any given fixed  $f \in W_h$ .

In general, we shall be concerned with the discretisation of finite element variational forms of general arity  $r + n > 0$ ,

$$a : V_h^1 \times V_h^2 \times \cdots \times V_h^r \times W_h^1 \times W_h^2 \times \cdots \times W_h^n \rightarrow \mathbb{R}, \quad (6)$$

defined on the product space  $V_h^1 \times V_h^2 \times \cdots \times V_h^r \times W_h^1 \times W_h^2 \times \cdots \times W_h^n$  of two sets  $\{V_h^j\}_{j=1}^r, \{W_h^j\}_{j=1}^n$  of discrete function spaces on  $\Omega$ . We refer to  $(v_1, v_2, \dots, v_r) \in V_h^1 \times V_h^2 \times \cdots \times V_h^r$  as *primary arguments*, and to  $(w_1, w_2, \dots, w_n) \in W_h^1 \times W_h^2 \times \cdots \times W_h^n$  as *coefficients* and write

$$a = a(v_1, \dots, v_r; w_1, \dots, w_n). \quad (7)$$

In the simplest case, all function spaces are equal but there are many important examples, such as mixed methods, where the arguments come from different function spaces. The choice of coefficient function spaces depends on the application; a polynomial basis simplifies exact integration, while in some cases evaluating coefficients in quadrature points may be required.

### 2.3 Discretisation

To discretise the form  $a$ , we introduce a set of bases  $\{\phi_i^1\}_{i=1}^{N^1}, \{\phi_i^2\}_{i=1}^{N^2}, \dots, \{\phi_i^r\}_{i=1}^{N^r}$  for the function spaces  $V_h^1, V_h^2, \dots, V_h^r$  respectively and let  $i = (i_1, i_2, \dots, i_r)$  be a multiindex of length  $|i| = r$ . The form  $a$  then defines a rank  $r$  tensor given by

$$A_i = a(\phi_{i_1}^1, \phi_{i_2}^2, \dots, \phi_{i_r}^r; w_1, w_2, \dots, w_n) \quad \forall i \in \mathcal{I}, \quad (8)$$

where  $\mathcal{I}$  is the index set

$$\begin{aligned} \mathcal{I} &= \prod_{j=1}^r [1, |V_h^j|] \\ &= \{(1, 1, \dots, 1), (1, 1, \dots, 2), \dots, (N^1, N^2, \dots, N^r)\}. \end{aligned} \quad (9)$$

We refer to the tensor  $A$  as the *discrete operator* generated by the form  $a$  and the particular choice of basis functions. For any given form of arity  $r + n$ , the tensor  $A$  is a (typically sparse) tensor of rank  $r$  and dimension  $|V_h^1| \times |V_h^2| \times \cdots \times |V_h^r| = N^1 \times N^2 \times \cdots \times N^r$ .

Typically, the rank  $r$  is 0, 1, or 2. When  $r = 0$ , the tensor  $A$  is a scalar (a tensor of rank zero), when  $r = 1$ , the tensor  $A$  is a vector (the ‘load vector’) and when  $r = 2$ , the tensor  $A$  is a matrix (the ‘stiffness matrix’). Forms of higher rank also appear, though they are rarely assembled as a higher-dimensional sparse tensor.

Note here that we consider the functions  $w_1, w_2, \dots, w_n$  as fixed in the sense that the discrete operator  $A$  is computed for a given set of functions, which we refer to as *coefficients*. As an example, consider again the variational problem (2) for Poisson’s equation. For the trilinear form  $a$ , the rank is  $r = 2$  and the number of coefficients is  $n = 1$ , while for the linear form  $L$ , the rank is  $r = 1$  and the number of coefficients is  $n = 1$ . We may also choose to directly compute the *action* of the form  $a$  obtained by assembling a vector from the form

$$a(v_1; w_1, w_2) = \int_{\Omega} w_1 \nabla v_1 \cdot \nabla w_2 \, dx, \quad (10)$$

where now  $r = 1$  and  $n = 2$ .

We list below a few other examples to illustrate the notation.

**Example 2.1:** Our first example is related to the divergence constraint in fluid flow. Let the form  $a$  be given by

$$a(q, u) = \int_{\Omega} q \nabla \cdot u \, dx, \quad q \in V_h^1, \quad u \in V_h^2, \quad (11)$$

where  $V_h^1$  is a space of scalar-valued functions and where  $V_h^2$  is a space of vector-valued functions. The form  $a : V_h^1 \times V_h^2 \rightarrow \mathbb{R}$  has two primary arguments and thus  $r = 2$ . Furthermore, the form does not depend on any coefficients and thus  $n = 0$ .

**Example 2.2:** Another common form in fluid flow (with variable density) is

$$a(v, u; w, \varrho) = \int_{\Omega} \varrho (w \cdot \nabla u) \cdot v \, dx. \quad (12)$$

Here,  $v \in V_h^1, u \in V_h^2, w \in W_h^1, \varrho \in W_h^2$ , where  $V_h^1, V_h^2$ , and  $W_h^1$  are spaces of vector-valued functions, while  $W_h^2$  is a space of scalar-valued functions. The form takes four arguments, where two of the arguments are coefficients,

$$a : V_h^1 \times V_h^2 \times W_h^1 \times W_h^2 \rightarrow \mathbb{R}. \quad (13)$$

Hence,  $r = 2$  and  $n = 2$ .

**Example 2.3:** We next consider the following form appearing in nonlinear convection-diffusion with a

power-law viscosity,

$$a(v; w, \mu, \varrho) = \int_{\Omega} \varrho(w \cdot \nabla w) \cdot v + \mu |\nabla w|^{2q} \nabla w : \nabla v \, dx. \quad (14)$$

Here,  $v \in V_h^1, w \in W_h^1, \mu \in W_h^2, \varrho \in W_h^3$ , where  $V_h^1$  and  $W_h^1$  are spaces of vector-valued functions, while  $W_h^2$  and  $W_h^3$  are spaces of scalar-valued functions. The form takes four arguments, where three of the arguments are coefficients,

$$a : V_h^1 \times W_h^1 \times W_h^2 \times W_h^3 \rightarrow \mathbb{R}. \quad (15)$$

Hence,  $r = 1$  and  $n = 3$ .

**Example 2.4:** The  $H^1(\Omega)$  norm of the error  $e = u - u_h$  squared is

$$a(; u, u_h) = \int_{\Omega} (u - u_h)^2 + |\nabla(u - u_h)|^2 \, dx. \quad (16)$$

The form takes two arguments and both are coefficients,

$$a : W_h^1 \times W_h^2 \rightarrow \mathbb{R}. \quad (17)$$

Hence,  $r = 0$  and  $n = 2$ .

Defining variational forms for coupled PDEs can be performed in two ways in the above described framework. One approach is to couple the variational forms on the linear algebra level, using block vectors and block matrices and defining one form for each block. Alternatively, a single form for the coupled system may be defined using mixed finite elements.

### 3 Finite element assembly

The standard algorithm for computing the global sparse tensor  $A$  is known as *assembly*, see Zienkiewicz et al. (1967–2005) and Hughes (1987). By this algorithm, the tensor  $A$  may be computed by assembling (summing) the contributions from the local entities of a finite element mesh. To express this algorithm for assembly of the global sparse tensor  $A$  for a general finite element variational form of rank  $r$ , we introduce the following notation and assumptions.

Let  $\mathcal{T} = \{K\}$  be a set of disjoint *cells* (a triangulation or tessellation) partitioning the domain  $\Omega = \cup_{K \in \mathcal{T}} K$ . Further, let  $\partial_e \mathcal{T}$  denote the set of *exterior facets* (the set of cell facets on the boundary  $\partial\Omega$ ), and let  $\partial_i \mathcal{T}$  denote the set of *interior facets* (the set of cell facets not on the boundary  $\partial\Omega$ ). For each discrete function space  $V_h^j$ ,  $j = 1, 2, \dots, r$ , we assume that the global basis  $\{\phi_i^j\}_{i=1}^{N_K^j}$  is obtained by patching together local function spaces  $\mathcal{P}_K^j$  on each cell  $K$  as determined by a local-to-global mapping.

We shall further assume that the variational form (6) may be expressed as a sum of integrals over the cells  $\mathcal{T}$ , the exterior facets  $\partial_e \mathcal{T}$  and the interior

facets  $\partial_i \mathcal{T}$ . We shall allow integrals expressed on disjoint subsets  $\mathcal{T} = \cup_{k=1}^{n_e} \mathcal{T}_k$ ,  $\partial_e \mathcal{T} = \cup_{k=1}^{n_e} \partial_e \mathcal{T}_k$  and  $\partial_i \mathcal{T} = \cup_{k=1}^{n_i} \partial_i \mathcal{T}_k$  respectively.

We thus assume that the form  $a$  is given by

$$\begin{aligned} a(v_1, \dots, v_r; w_1, \dots, w_n) &= \sum_{k=1}^{n_e} \sum_{K \in \mathcal{T}_k} \int_K I_k^c(v_1, \dots, v_r; w_1, \dots, w_n) \, dx \\ &+ \sum_{k=1}^{n_e} \sum_{S \in \partial_e \mathcal{T}_k} \int_S I_k^e(v_1, \dots, v_r; w_1, \dots, w_n) \, ds \\ &+ \sum_{k=1}^{n_i} \sum_{S \in \partial_i \mathcal{T}_k} \int_S I_k^i(v_1, \dots, v_r; w_1, \dots, w_n) \, ds. \end{aligned} \quad (18)$$

We refer to an integral over a cell  $K$  as a *cell integral*, an integral over an exterior facet  $S$  as an *exterior facet integral* (typically used to implement Neumann and Robin type boundary conditions), and to an integral over an interior facet  $S$  as an *interior facet integral* (typically used in discontinuous Galerkin methods).

For simplicity, we consider here initially assembly of the global sparse tensor  $A$  corresponding to a form  $a$  given by a single integral over all cells  $\mathcal{T}$ , and later extend to the general case where we must also account for contributions from several cell integrals, interior facet integrals and exterior facet integrals.

We thus consider the form

$$\begin{aligned} a(v_1, \dots, v_r; w_1, \dots, w_n) &= \sum_{K \in \mathcal{T}} \int_K I^c(v_1, \dots, v_r; w_1, \dots, w_n) \, dx, \end{aligned} \quad (19)$$

for which the global sparse tensor  $A$  is given by

$$A_i = \sum_{K \in \mathcal{T}} \int_K I^c(\phi_{i_1}^1, \dots, \phi_{i_r}^r; w_1, \dots, w_n) \, dx. \quad (20)$$

To see how to compute the tensor  $A$  by summing the local contributions from each cell  $K$ , we let  $n_K^j = |\mathcal{P}_K^j|$  denote the dimension of the local finite element space on  $K$  for the  $j$ th primary argument  $v_j \in V_h^j$  for  $j = 1, 2, \dots, r$ . Furthermore, let

$$\iota_K^j : [1, n_K^j] \rightarrow [1, N^j] \quad (21)$$

denote the local-to-global mapping for  $V_h^j$ , that is, on any given  $K \in \mathcal{T}$ , the mapping  $\iota_K^j$  maps the number of a local degree of freedom (or, equivalently, local basis function) to the number of the corresponding global degree of freedom (or, equivalently, global basis function). We then define for each  $K \in \mathcal{T}$  the collective local-to-global mapping  $\iota_K : \mathcal{I}_K \rightarrow \mathcal{I}$  by

$$\iota_K(i) = (\iota_K^1(i_1), \iota_K^2(i_2), \dots, \iota_K^r(i_r)) \quad \forall i \in \mathcal{I}_K, \quad (22)$$

where  $\mathcal{I}_K$  is the index set

$$\begin{aligned} \mathcal{I}_K &= \prod_{j=1}^r [1, |\mathcal{P}_K^j|] \\ &= \{(1, 1, \dots, 1), (1, 1, \dots, 2), \dots, (n_K^1, n_K^2, \dots, n_K^r)\}. \end{aligned} \quad (23)$$

Furthermore, for each  $V_h^j$  we let  $\{\phi_i^{K,j}\}_{i=1}^{n_K^j}$  denote the restriction to an element  $K$  of the subset of the basis  $\{\phi_i^j\}_{i=1}^{N^j} \subset \mathcal{P}_K^j$  of  $V_h^j$  supported on  $K$ .

We may now compute  $A$  by summing the contributions from the local cells,

$$\begin{aligned} A_i &= \sum_{K \in \mathcal{T}_i} \int_K I^c(\phi_{i_1}^1, \dots, \phi_{i_r}^r; w_1, \dots, w_n) dx \\ &= \sum_{K \in \mathcal{T}_i} \int_K I^c(\phi_{(\iota_K^1)^{-1}(i_1)}^{K,1}, \dots, \phi_{(\iota_K^r)^{-1}(i_r)}^{K,r}; w_1, \dots, w_n) dx \\ &= \sum_{K \in \mathcal{T}_i} A_{\iota_K^{-1}(i)}^K, \end{aligned} \quad (24)$$

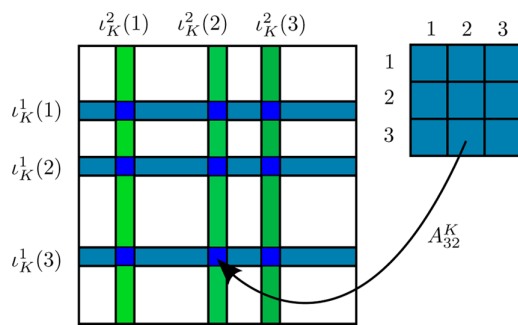
where  $A^K$  is the local *cell tensor* on cell  $K$  (the ‘element stiffness matrix’), given by

$$A_i^K = \int_K I^c(\phi_{i_1}^{K,1}, \dots, \phi_{i_r}^{K,r}; w_1, \dots, w_n) dx, \quad (25)$$

and where  $\mathcal{T}_i$  denotes the set of cells on which all basis functions  $\phi_{i_1}^1, \phi_{i_2}^2, \dots, \phi_{i_r}^r$  are supported. Similarly, we may sum the local contributions from the exterior and interior facets in the form of local *exterior facet tensors* and *interior facet tensors*.

In Algorithm 1, we present a general algorithm for assembling the contributions from the local cell, exterior facet and interior facet tensors into a global sparse tensor. In all cases, we iterate over all entities (cells, exterior or interior facets), compute the local cell tensor  $A^K$  (or exterior/interior facet tensor  $A^S$ ) and add it to the global sparse tensor as determined by the local-to-global mapping, see Figure 1.

**Figure 1** Adding the entries of a cell tensor  $A^K$  to the global tensor  $A$  using the local-to-global mapping  $\iota_K$ , illustrated here for a rank two tensor (a matrix) (see online version for colours)



#### 4 Software framework for finite element assembly

In a finite element application code, typical input from the user is the variational (weak) form of a PDE, a choice of finite elements, a geometry represented by a mesh, and user-defined functions that appear as coefficients in the variational form. For a linear PDE, the typical solution procedure consists of first assembling a (sparse) linear system  $AU = b$  from given user input and then solving

that linear system to obtain the degrees of freedom  $U$  for the discrete finite element approximation  $u_h$  of the exact solution  $u$  of the PDE. Even when the solution procedure is more involved, as for a nonlinear problem requiring an iterative procedure, each iteration may involve assembling matrices and vectors. It is therefore clear that the assembly of matrices and vectors (or in general tensors) is an important task for any finite element software framework. We refer to the software component responsible for assembling a global tensor from given user input consisting of a variational form, finite element function spaces, mesh and coefficients as the *Assembler*.

As demonstrated in Algorithm 1, the Assembler needs to iterate over the cells in the mesh, tabulate degree of freedom mappings, extract local values of coefficients, compute the local element tensor, and add each element tensor to the global tensor which is the final output. Thus, the Assembler is a software component where many other components are combined. It is therefore important that the software components on which the Assembler depends have well-defined interfaces. We discuss some issues relating to the design of these software components below and then demonstrate how these software components together with the Assembler may be combined into a general software framework for finite element assembly.

##### 4.1 Variational forms

Implementations of discrete variational forms in a general finite element library usually consist of programming expressions for the integrands  $I_k^c$ ,  $I_k^e$ ,  $I_k^i$  (see equation (18)), eventually writing a quadrature loop and a loop over element matrix indices depending on the abstraction level of the library (see Bangerth et al., 2007; Langtangen, 2003b). An alternative approach is to apply exact integration instead of quadrature. In either case, the result of this computation may be communicated through the UFC interface.

The motivation behind the UFC interface is to separate the implementation of the form from other details of the assembly such as the mesh and the linear algebra libraries in use.

In the FEniCS finite element software framework, a high-level form language embedded in Python is used to define the variational form and finite elements. This reduces the distance from the mathematical formulation of a PDE to an implementation of a PDE solver, removes tedious and error-prone tasks such as coding PDE-specific assembly loops, and enables rapid prototyping of new models and methods. To retain computational efficiency, we generate efficient low-level code from the abstract form description, using exact integration where possible. Code generation adds another complexity layer to the software, and it becomes even more important to keep a clear separation between software components such that the interface between generated code and library code is well defined.

This is achieved by generating implementations of the UFC interface.

---

**Algorithm 1** Assembling the global tensor  $A$  from the local contributions on all cells, exterior and interior facets. For assembly over exterior facets,  $K(S)$  refers to the cell  $K \in \mathcal{T}$  incident with the exterior facet  $S$ , and for assembly over interior facets,  $K(S)$  refers to the “macro cell” consisting of the pair of cells  $K^+$  and  $K^-$  incident with the interior facet  $S$ .

---

$A = 0$

(i) *Assemble contributions from all cells*  
**for each**  $K \in \mathcal{T}$

**for**  $j = 1, 2, \dots, r$ :  
         Tabulate the local-to-global mapping  $\iota_K^j$

**for**  $j = 1, 2, \dots, n$ :  
         Extract the values of  $w_j$  on  $K$

    Take  $0 \leq k \leq n_c$  such that  $K \in \mathcal{T}_k$   
     Tabulate the cell tensor  $A^K$  for  $I_k^c$   
     Add  $A_i^K$  to  $A_{\iota_K^1(i_1), \iota_K^2(i_2), \dots, \iota_K^r(i_r)}$  for  $i \in I_K$

(ii) *Assemble contributions from all exterior facets*  
**for each**  $S \in \partial_e \mathcal{T}$

**for**  $j = 1, 2, \dots, r$ :  
         Tabulate the local-to-global mapping  $\iota_{K(S)}^j$

**for**  $j = 1, 2, \dots, n$ :  
         Extract the values of  $w_j$  on  $K(S)$

    Take  $0 \leq k \leq n_e$  such that  $S \in \partial_e \mathcal{T}_k$   
     Tabulate the exterior facet tensor  $A^S$  for  $I_k^e$   
     Add  $A_i^S$  to  $A_{\iota_{K(S)}^1(i_1), \iota_{K(S)}^2(i_2), \dots, \iota_{K(S)}^r(i_r)}$  for  $i \in I_{K(S)}$

(iii) *Assemble contributions from all interior facets*  
**for each**  $S \in \partial_i \mathcal{T}$

**for**  $j = 1, 2, \dots, r$ :  
         Tabulate the local-to-global mapping  $\iota_{K(S)}^j$

**for**  $j = 1, 2, \dots, n$ :  
         Extract the values of  $w_j$  on  $K(S)$

    Take  $0 \leq k \leq n_i$  such that  $S \in \partial_i \mathcal{T}_k$   
     Tabulate the interior facet tensor  $A^S$  for  $I_k^i$   
     Add  $A_i^S$  to  $A_{\iota_{K(S)}^1(i_1), \iota_{K(S)}^2(i_2), \dots, \iota_{K(S)}^r(i_r)}$  for  $i \in I_{K(S)}$

---

## 4.2 Mesh libraries

Many different representations of computational meshes exist. Typically, each finite element library provides its own internal implementation of a computational mesh. We do not wish to tie the UFC interface to one particular mesh representation or one particular library. Still, several operations like the element tensor computation depends on local mesh data. For this reason, the UFC interface provides a low-level data structure to communicate single cell data. In addition, a small data structure is used to communicate global mesh dimensions which are necessary for computing the local-to-global mapping. Assemblers implemented on top of the UFC

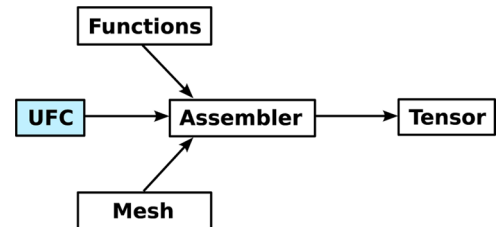
interface must therefore be able to copy/translate cell data from the mesh library being used to the UFC data structure (involving a minimal overhead). This makes it possible to achieve separation between the mesh representation and the element tensor computation. The Assembler implemented in FEniCS (as part of DOLFIN) is implemented for one particular mesh format, see Logg (2009), but an Assembler component could easily be written for other mesh libraries like the PETSc *Sieve*, see Knepley and Karpeev (2009) and the DUNE-Grid interface (Bastian et al., 2008a, 2008b).

## 4.3 Linear algebra libraries

It is desirable to reuse existing high-performance linear algebra libraries like PETSc and Trilinos. It is therefore important that the Assembler is able to assemble element tensors into global matrices and vectors implemented by external libraries. Aggregation into matrix and vector data structures are fairly similar operations, typically consisting of passing array pointers to existing functions in the linear algebra libraries. By implementing a common interface for assembly into tensors of arbitrary rank, the same assembly routine can be reused for any linear algebra library without changes. This avoids duplication of assembly code, and one may easily change the output format of the assembly procedure. The details of these interfaces are beyond the scope of the current paper. At the time of writing, we have written assembly routines with support for matrices and vectors from Epetra (Trilinos), PETSc, PyCC, and uBLAS in addition to scalars.

With components available for finite element variational forms, mesh representation, and linear algebra, we may use the UFC interface to combine these components to build a PSE for partial differential equations. The central component of this PSE is the Assembler. As illustrated in Figure 2, the Assembler takes as input a variational form, communicated through the UFC interface, a mesh and a set of functions (the coefficients), and assembles a tensor.

**Figure 2** Assembling a tensor from a given UFC, mesh and functions (coefficients) (see online version for colours)



## 4.4 High-level interfaces

In FEniCS, we have additional application-level abstractions for expressing variational forms, meshes, functions and linear algebra objects to achieve a consistent high-level user-interface. The generation of



the UFC code may then be hidden from the user, who just provides a high-level description of the form. The PSE may then automatically generate the UFC at run-time, functioning as Just-In-Time (JIT) compiler, and call the Assembler with the generated UFC. Below, we demonstrate how this may be done in the Python interface of DOLFIN. The user here defines a finite element function space, and a pair of bilinear and linear forms  $a(v, u) = \int_{\Omega} \nabla v \cdot \nabla u + vu \, dx$  and  $L = \int_{\Omega} v f \, dx$ , from which a matrix and vector may be assembled by calls to the function `assemble`. A linear system solver may then be invoked to compute the degrees of freedom  $U$  of the solution.

```

element = FiniteElement("CG", "triangle", 1)

v = TestFunction(element)
u = TrialFunction(element)
f = Function(element, mesh, 100.0)

a = dot(grad(v), grad(u))*dx + v*u*dx
L = v*f*dx

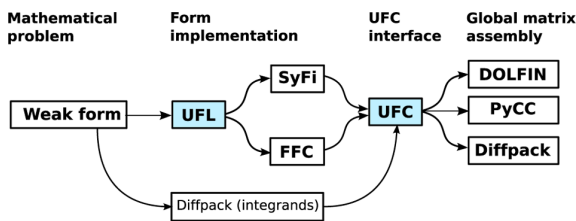
A = assemble(a, mesh)
b = assemble(L, mesh)

U = solve(A, b)

```

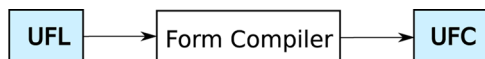
While FEniCS provides an integrated environment, including the PSE DOLFIN and the two form compilers FFC (FEniCS Form Compiler) and SFC (SyFi Form Compiler), the fact that these components comply with the UFC interface means that they may also be used interchangeably in heterogeneous environments together with other libraries (that implement or use the UFC interface). This is illustrated in the flow diagram of Figure 3 where alternate routes from mathematical description to matrix assembly are demonstrated.

**Figure 3** Alternate routes from mathematical description to matrix assembly enabled by the UFC interface (Note that the Diffpack example is fictional) (see online version for colours)



In Figure 3, we have also included another interface UFL (Unified Form Language) which provides a unified way to express finite element variational forms. The UFL interface is currently in development. Together, UFL and UFC provide a unified interface for the input and output of form compilers, see Figure 4.

**Figure 4** An abstract definition (UFL) of a finite element variational form is given as input to a form compiler, which generates UFC code as output (see online version for colours)



## 5 The UFC interface

The UFC interface consists of a small collection of abstract C++ classes that represent common components for assembling tensors using the finite element method. These classes are accompanied by a well documented (Alnæs et al., 2008) set of conventions for numbering of cell data and other arrays. We have strived to make the classes as simple as possible while not sacrificing generality or efficiency. Data is passed as plain C arrays for efficiency and minimal dependencies. Most functions are declared *const*, reflecting that the operations they represent should not change the outcome of future operations.<sup>2</sup> Other initialisation of implementation-specific data should ideally be performed in constructors.

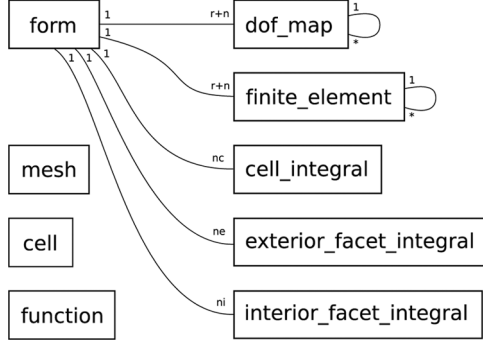
One can ask why the UFC interface consists of classes and not plain functions. There are three reasons for this. First, we want to handle each form as a self contained ‘black box’, which can be passed around easily in an application. Many functions belong together conceptually, thus making it natural to collect them in a class ‘namespace’. Second, we need multiple versions of each function in the software representation of variational forms, in particular to represent multiple variational forms and multiple finite element function spaces. This is best achieved by making each such function a member function of a class and having multiple instances of that class. Third, UFC function implementations may need access to stored data, and with a plain function-based interface these data would then need to be global variables. In particular, when existing libraries or applications want to implement the UFC interface, it may be necessary for the subclasses of UFC classes to inherit from existing classes or to have pointers to other objects.

### 5.1 Class relations

Figure 5 shows all the classes and their relations. The classes `mesh`, `cell`, and `function` provide the means for communicating mesh and coefficient function data as arguments. Each argument of the form (both primary arguments and coefficients) is represented by a `finite_element` and `dof_map` object. The integrals are represented by one of the classes `cell_integral`, `exterior_facet_integral`, or `interior_facet_integral`. An object of the class `form` gives access to all other objects in a particular implementation. In this paper, we will not describe all the functions of these classes in detail. A complete specification can be found in the manual (Alnæs et al., 2008).

At the core of UFC is the class `form`, which represents the general variational form  $a$  of equation (18). Subclasses of `form` must implement factory functions which may be called to create `cell_integral`, `exterior_facet_integral` and `interior_facet_integral` objects. These objects in



**Figure 5** UML diagram of the UFC class relations

turn know how to compute their respective contribution from a cell or facet during assembly. A code fragment from the *form* class declaration is shown below.

```
class form
{
public:
    ...

    /// Create cell integral on sub domain i
    virtual cell_integral*
    create_cell_integral(unsigned int i)
    const = 0;

    /// Create exterior facet integral on sub domain i
    virtual exterior_facet_integral*
    create_exterior_facet_integral(unsigned int i)
    const = 0;

    /// Create interior facet integral on sub domain i
    virtual interior_facet_integral*
    create_interior_facet_integral(unsigned int i)
    const = 0;
};
```

The *form* class also specifies functions for creating *finite\_element* and *dof\_map* objects for the finite element function spaces  $\{V_h^j\}_{j=1}^r, \{W_h^j\}_{j=1}^n$  of the variational form. The *finite\_element* object provides functionality such as evaluation of degrees of freedom and evaluation of basis functions and their derivatives. The *dof\_map* object provides functionality such as tabulating the local-to-global mapping of degrees of freedom on a single element, as well as tabulation of subsets associated with particular mesh entities, used to apply Dirichlet boundary conditions and build connectivity information.

Both the *finite\_element* and *dof\_map* classes can represent mixed elements, in which case it is possible to obtain *finite\_element* and *dof\_map* objects for each sub-element in a hierarchical manner. Vector elements composed of scalar elements are in this context seen as special cases of mixed elements where all sub-elements are equal. Thus, e.g., from a *dof\_map* representing a  $P_2 - P_1$  Taylor-Hood element, it is possible to extract one *dof\_map* for the quadratic vector element and one *dof\_map* for the linear scalar element. From the vector element, a *dof\_map* for the quadratic scalar element of each vector

component can be obtained. This can be used to access subcomponents from the solution of a mixed system.

## 5.2 Stages in the assembly algorithm

Next, we focus on a few key parts of the interface and explain how these can be used to implement the assembly algorithm (Algorithm 1). This algorithm consists of three stages:

- i assembling the contributions from all cells
- ii assembling the contributions from all exterior facets
- iii assembling the contributions from all interior facets.

Each of the three assembly stages (i)–(iii) of Algorithm 1 is further composed of five steps. In the first step, the polygon  $K$  is fetched from the mesh, typically implemented by filling a *cell* structure (see Figure 6) with coordinate data and global numbering of the mesh entities in the cell. This step depends on the specific mesh being used.

**Figure 6** Data structure for communicating single cell data

```
enum shape {interval, triangle, quadrilateral,
            tetrahedron, hexahedron};

class cell {
public:
    shape cell_shape;
    unsigned int topological_dimension;
    unsigned int geometric_dimension;

    /// Array of global indices for the mesh entities of the cell
    unsigned int** entity_indices;

    /// Array of coordinates for the vertices of the cell
    double** coordinates;
};
```

Secondly, the local-to-global mapping of degrees of freedom is tabulated for each of the function spaces. That is, for each of the discrete finite element spaces on  $K$ , we tabulate (or possibly compute) the global indices for the degrees of freedom on  $\{V_h^j\}_{j=1}^r$  and  $\{W_h^j\}_{j=1}^n$ .

The class *dof\_map* represents the mapping between local and global degrees of freedom for a finite element space. A *dof\_map* is initialised with global mesh dimensions by calling the function *init\_mesh*(const mesh& m). If this function returns true, the *dof\_map* should be additionally initialised by calling the function *init\_cell*(const mesh& m, const cell& c) for each cell in the global mesh, followed by *init\_cell\_finalize* after the last cell. After the initialisation stage, the mapping may be tabulated at a given cell by calling a function with the following signature.

```
void dof_map::tabulate_dofs(unsigned int* dofs,
                           const mesh& m,
                           const cell& c) const
```

Here, `unsigned int* dofs` is a pointer to the first element of an array of unsigned integers that will be filled with the local-to-global mapping on the current cell during the function call.

In the third step of each stage of Algorithm 1, we may use the tabulated local-to-global mapping to interpolate (extract) the local values of any of the coefficients in  $\{W_h^j\}_{j=1}^n$ .

If a coefficient  $w_j$  is not given as a linear combination of basis functions for  $W_h^j$ , it must at this step be interpolated into  $W_h^j$ , using the interpolant defined by the degrees of freedom of  $W_h^j$  (for example point evaluation at a set of nodal points). In this case, the coefficient function is passed as an implementation of the function interface (a simple functor) to the function `evaluate_dofs`.

```
/// Evaluate linear functionals for all dofs on the function f
void finite_element::evaluate_dofs(double *values,
                                   const function& f,
                                   const cell& c) const
```

In the fourth step, the local element tensor contributions (cell or exterior/interior facet tensors) are computed. This is done by a call to the function `tabulate_tensor`, illustrated below for a cell integral.

```
void cell_integral::tabulate_tensor(double* A,
                                   const double * const * w,
                                   const cell& c) const
```

Similarly, one may evaluate interior and exterior facet contributions using slightly different function signatures.

Finally, at the fifth step, the local element tensor contributions are added to the global tensor, using the local-to-global mappings previously obtained by calls to the `tabulate_dofs` function. This is a simple operation that depends on the linear algebra library in use.

## 6 Examples

In this section, we demonstrate how UFC is used in practice in DOLFIN, FFC, and SFC. First, we show a part of the assembly algorithm (Algorithm 1) as implemented in DOLFIN. We then show examples of input to the form compilers FFC and SFC as well as part of the corresponding UFC code generated as output. Examples include Poisson's equation and linear convection (see Example 2.2).

### 6.1 An example UFC assembler

To demonstrate how one may implement an assembler based on the UFC interface, we provide here a (somewhat simplified) excerpt from the DOLFIN assembler.<sup>3</sup>

```
for (CellIterator cell(mesh); !cell.end(); ++cell)
{
    ufc.update(*cell);

    for (uint i = 0; i < ufc.form.rank(); i++)
        dof_map_set[i].tabulate_dofs(ufc.dofs[i], *cell);

    for (uint i = 0; i < coefficients.size(); i++)
        coefficients[i]->interpolate(ufc.w[i], ufc.cell,
                                     *ufc.coefficient_elements[i],
                                     *cell);

    integral->tabulate_tensor(ufc.A, ufc.w, ufc.cell);

    A.add(ufc.A, ufc.local_dimensions, ufc.dofs);
}
```

The outer loop iterates over all cells in a given mesh. For each cell, a `ufc::cell` is updated and the local-to-global mapping is constructed. We then interpolate all the form coefficients on the cell and compute the element tensor. At the end of the iteration, the local-to-global mapping is used to add the local tensor to the global tensor.

### 6.2 FFC examples

The form compiler FFC provides a simple language for specification of variational forms, which may be entered either directly in Python or in text files given to the compiler on the command-line. For each variational form given as input, FFC generates UFC-compliant C++ code for evaluation of the corresponding element tensor(s).

#### Poisson's equation

We consider the following input file to FFC for Poisson's equation.

```
element = FiniteElement("Lagrange", "triangle", 1)

v = TestFunction(element)
u = TrialFunction(element)
f = Function(element)

a = dot(grad(v), grad(u))*dx
L = v*f*dx
```

Here, two forms `a` (bilinear) and `L` (linear) are defined. Both the test and trial spaces are spanned by linear Lagrange elements on triangles in two dimensions. When compiling this code using FFC, a C++ header file is created, containing UFC code that may be used to assemble the global sparse stiffness matrix and load vector. Below, we present the code generated for evaluation of the element stiffness matrix for the bilinear form `a`.

In FFC, an element tensor contribution is computed as a tensor contraction between a geometry tensor varying from cell to cell, and a geometry independent tensor on a reference element, see Kirby and Logg (2006, 2007). For simple forms, like the one under discussion, the main work is then to construct the geometry tensor,

related to the geometrical mapping between the reference element and physical element.

```
virtual void tabulate_tensor(double* A,
                           const double * const * w,
                           const ufc::cell& c) const
{
    // Extract vertex coordinates
    const double * const * x = c.coordinates;

    // Compute Jacobian of affine map from reference cell
    const double J_00 = x[1][0] - x[0][0];
    const double J_01 = x[2][0] - x[0][0];
    const double J_10 = x[1][1] - x[0][1];
    const double J_11 = x[2][1] - x[0][1];

    // Compute determinant of Jacobian
    double detJ = J_00*J_11 - J_01*J_10;

    // Compute inverse of Jacobian
    const double Jinv_00 = J_11 / detJ;
    const double Jinv_01 = -J_01 / detJ;
    const double Jinv_10 = -J_10 / detJ;
    const double Jinv_11 = J_00 / detJ;

    // Set scale factor
    const double det = std::abs(detJ);

    // Compute geometry tensors
    const double G0_0_0 = det*(Jinv_00*Jinv_00 + Jinv_01*Jinv_01);
    const double G0_0_1 = det*(Jinv_00*Jinv_10 + Jinv_01*Jinv_11);
    const double G0_1_0 = det*(Jinv_10*Jinv_00 + Jinv_11*Jinv_01);
    const double G0_1_1 = det*(Jinv_10*Jinv_10 + Jinv_11*Jinv_11);

    // Compute element tensor
    A[0] = 0.5*G0_0_0 + 0.5*G0_0_1 + 0.5*G0_1_0 + 0.5*G0_1_1;
    A[1] = -0.5*G0_0_0 - 0.5*G0_1_0;
    A[2] = -0.5*G0_0_1 - 0.5*G0_1_1;
    A[3] = -0.5*G0_0_0 - 0.5*G0_0_1;
    A[4] = 0.5*G0_0_0;
    A[5] = 0.5*G0_0_1;
    A[6] = -0.5*G0_1_0 - 0.5*G0_1_1;
    A[7] = 0.5*G0_1_0;
    A[8] = 0.5*G0_1_1;
}
```

Having computed the element tensor, one needs to compute the local-to-global mapping in order to know where to insert the local contributions in the global tensor. This mapping may be obtained by calling the member function `tabulate_dofs` of the class `ufc::dof_map`. FFC uses an implicit ordering scheme, based on the indices of the topological entities in the mesh. This information may be extracted from the `ufc::cell` attribute `entity_indices`.

```
virtual void tabulate_dofs(unsigned int* dofs,
                          const ufc::mesh& m,
                          const ufc::cell& c) const
{
    dofs[0] = c.entity_indices[0][0];
    dofs[1] = c.entity_indices[0][1];
    dofs[2] = c.entity_indices[0][2];
}
```

For Lagrange elements on triangles, each degree of freedom is associated with a global vertex. Hence, FFC constructs the mapping by picking the corresponding global vertex number for each degree of freedom.

### Linear convection

Consider the variational form in Example 2.2. The input file to FFC reads as follows.

```
vector_element = VectorElement("Lagrange", "triangle", 1)
scalar_element = FiniteElement("Lagrange", "triangle", 1)

v = TestFunction(vector_element)
u = TrialFunction(vector_element)
w = Function(vector_element)
rho = Function(scalar_element)

a = rho*v[i]*w[j]*u[i].dx(j)*dx
```

The code generated for the `tabulate_tensor` function is presented below. Computations involving coefficients are performed by interpolating the functions  $w$  and  $\rho$  on the cell under consideration. These values are stored in the array `w` below. For clarity, some code has been omitted in this example.

```
virtual void tabulate_tensor(double* A,
                           const double * const * w,
                           const ufc::cell& c) const
{
    // Extract vertex coordinates and compute Jacobian etc
    // as in previous example
    ...
    // Compute coefficients
    const double c1_0_0_0 = w[1][0];
    const double c1_0_0_1 = w[1][1];
    const double c1_0_0_2 = w[1][2];
    const double c0_0_1_0 = w[0][0];
    ...
    const double c0_0_1_5 = w[0][5];
    // Compute geometry tensors
    const double G0_0_0_0 = det*c1_0_0_0*c0_0_1_0*Jinv_00;
    const double G0_0_0_1 = det*c1_0_0_0*c0_0_1_0*Jinv_10;
    const double G0_0_1_0 = det*c1_0_0_0*c0_0_1_1*Jinv_00;
    ...
    // Compute element tensor
    A[0] = -0.05*G0_0_0_0 - ...
    A[1] = 0.05*G0_0_0_0 + ...
    A[2] = 0.05*G0_0_0_1 + ...
    ...
}
```

The local-to-global mapping for the space of piecewise linear vectors is computed by associating two values with each vertex. The code generated for `tabulate_dofs` is presented below.

```
virtual void tabulate_dofs(unsigned int* dofs,
                          const ufc::mesh& m,
                          const ufc::cell& c) const
{
    dofs[0] = c.entity_indices[0][0];
    dofs[1] = c.entity_indices[0][1];
    dofs[2] = c.entity_indices[0][2];
    unsigned int offset = m.num_entities[0];
    dofs[3] = offset + c.entity_indices[0][0];
    dofs[4] = offset + c.entity_indices[0][1];
    dofs[5] = offset + c.entity_indices[0][2];
}
```

FFC generates code for arbitrary multilinear forms and currently supports arbitrary degree continuous Lagrange elements, discontinuous elements, RT elements, BDM elements, BDFM elements and Nedelec elements in two and three space dimensions.

### 6.3 SFC examples

SFC is another form compiler producing UFC code, in which the user defines variational forms in Python using a symbolic engine based on GiNaC (Bauer et al., 2006). It has a slightly different feature set than FFC, such as using symbolic differentiation to automatically compute

the Jacobi matrix of a nonlinear form. The resulting low-level UFC code is very similar.

### Power-law viscosity

Example 2.3 is specified in SFC as follows.

```
vector_element = VectorElement("Lagrange", "triangle", 1)
scalar_element = FiniteElement("DG", "triangle", 0)

v = TestFunction(vector_element)
w = Function(vector_element)
mu = Function(scalar_element)
rho = Function(scalar_element)

def power_law(v, w, mu, rho, itg):
    q = 0.3
    GinvT = itg.GinvT()
    Dw = grad(w, GinvT)
    Dv = grad(v, GinvT)
    wDw = dot(w, Dw)

    return rho*dot(wDw, v) + mu*inner(Dw,Dw)**q * inner(Dv,Dv)

F_form = Form(basisfunctions = [v],
              coefficients = [w, mu, rho])
F_form.add_cell_integral(power_law)
J_form = Jacobi(F_form)
```

The syntax for defining elements and arguments is the same as in FFC, but the integrand is specified in a slightly different syntax.<sup>4</sup> This code also computes the form corresponding to the Jacobian matrix, using symbolic differentiation. The generated code for computing the Jacobian matrix is in this case more complicated but it implements the UFC interface in the same manner as in the previous examples.

In the current implementation, SFC explicitly constructs a local-to-global mapping at run-time. In this case, with Lagrange elements, the global coordinates identify the degrees of freedom. The UFC interface supports constructing the local-to-global mapping through the `init_mesh` and `init_cell` methods of `ufc::dof_map`. Below, we present the code generated for `init_cell` (where we use additional structures of type `Ptv` (point) for representing degrees of freedom and the container `Dof_Ptv` `dof` for building the local-to-global mapping).

```
void dof_map_2D::init_cell(const ufc::mesh& m, const ufc::cell& c)
{
    // coordinates
    double x0 = c.coordinates[0][0]; double y0 = c.coordinates[0][1];
    double x1 = c.coordinates[1][0]; double y1 = c.coordinates[1][1];
    double x2 = c.coordinates[2][0]; double y2 = c.coordinates[2][1];

    // affine map
    double G00 = x1 - x0;
    double G01 = x2 - x0;

    double G10 = y1 - y0;
    double G11 = y2 - y0;

    unsigned int element = c.entity_indices[2][0];

    double dof0[2] = { x0, y0 };
    Ptv pdof0(2, dof0);
    dof.insert_dof(element, 0, pdof0);

    double dof1[2] = { G00+x0, y0+G10 };
    Ptv pdof1(2, dof1);
    dof.insert_dof(element, 1, pdof1);

    double dof2[2] = { x0+G01, G11+y0 };
    Ptv pdof2(2, dof2);
    dof.insert_dof(element, 2, pdof2);
}
```

The `dof_map` class is only responsible for the uniqueness of the local-to-global mapping. Possible renumbering strategies may be imposed by the assembler, for example to minimise communication when assembling in parallel.

## 7 Discussion

We have used (generated) UFC for many applications, including Poisson's equation; convection–diffusion–reaction equations; continuum equations for linear elasticity, hyperelasticity, and plasticity; the incompressible Navier–Stokes equations; and mixed formulations for the Hodge Laplacian. The types of finite elements involved include standard continuous Lagrange elements of arbitrary order, discontinuous Galerkin formulations, BDM elements, Raviart–Thomas elements, Crouzeix–Raviart elements, and Nedelec elements.

The form compilers FFC and SFC are UFC compliant, both generating efficient UFC code from an abstract problem definition. Assemblers have been implemented in DOLFIN and PyCC, using the DOLFIN mesh representation, and together covering linear algebra formats from PETSc, Trilinos (Epetra), uBLAS, and PyCC. Parallel assembly is supported in the current development version of DOLFIN, without requiring any modifications to UFC since it operates on an element level. Altogether, this demonstrates that the UFC interface is flexible both in terms of the applications and finite element formulations it covers, and in terms of its interoperability with existing libraries.

One of the main limitations in the current version of the UFC interface (v1.1) is the assumption of a homogeneous mesh, that is, only one cell shape is allowed throughout the mesh. Thus, although mesh ordering conventions have been defined for the interval, triangle, tetrahedron, quadrilateral, and hexahedron, only one type of shape can be used at any time. Also, higher order (non-affinely mapped) meshes are not supported in the current version of the interface. Another limitation is that only one fixed finite element space can be chosen for each argument of the form, which excludes  $p$ -refinement (increasing the element order in a subset of the cells). All these limitations may be removed in future versions of UFC, and we encourage interested developers to make contact to address these limitations.

UFC provides a unified interface for code generated as *output* by form compilers such as FFC and SFC. Similarly, we are currently working on a specification for a Unified Form Language (UFL) to function as a common *input* to form compilers. Currently, both FFC and SFC provide (different) form languages for easy specification of variational forms in a high-level syntax. With a UFL, a user may specify a variational form in that language and assemble the corresponding discrete operator (tensor), independently of the components being used to generate the UFC code from the UFL, and independently of the components being used to assemble the tensor from the UFC form.

## 8 Conclusion

We have presented a general framework for assembly of finite element variational forms. Based on this framework, we have then extracted an interface (UFC) that may be used to provide a communication layer between general-purpose and problem-specific code for assembly of finite element variational forms.

The interface makes minimal assumptions on the type of problem being solved and the data structures involved. For example, the discrete variational form may in general be multilinear and hence assemble into a tensor of arbitrary rank. The basic data structures used to pass data through the interface are composed of plain C arrays. The minimal set of assumptions on problem and data structures enables application of the interface to a wide range of variational forms and a large collection of finite element libraries.

We have used the UFC interface in the implementation of the FEniCS suite of finite element tools. In a simple Python script, one may define a variational form and a mesh, and assemble the corresponding global sparse matrix (or vector). When doing so, UFC code is generated by either of the form compilers FFC or SFC, and passed to the UFC-compatible assembler of the general-purpose finite element library DOLFIN.

We encourage developers of finite element software to use the UFC interface in their libraries. By doing so, those libraries may directly take advantage of the form compilers FFC or SFC to specify finite element problems. Moreover, one can think of already existing specifications of complicated finite element problems that via UFC can be combined with other libraries than the specifications were originally written for. We have tried to make minimal assumptions to make this possible.

We believe that UFC itself and the ideas behind it constitute an important step towards greater flexibility in finite element software. By code generation via tools like FFC and SFC, this flexibility may be retained also in combination with very high performance.

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## Notes

<sup>1</sup>It is assumed that any given function may be represented (exactly or approximately) in some finite element space. Alternatively, functions may be approximated by quadrature. Quadrature representation is not discussed here, but is covered by the UFC specification and implemented by the form compilers FFC and SFC.

<sup>2</sup>The exceptions are the functions to initialise a *dof\_map*.

<sup>3</sup>The *ufc* object is here an instance of a simple DOLFIN class that holds pointers to arrays and UFC container classes, such as the array *A* and cell data *ufc::cell*, needed to communicate through the UFC interface.

<sup>4</sup>The variable *itg* is an integral object containing information about the mapping between physical coordinates and the reference element.

## Websites

Cactus, <http://www.cactuscode.org/>  
 COMSOL Multiphysics, <http://www.comsol.com>  
 FEniCS, <http://www.fenics.org>  
 Getfem++, <http://home.gna.org/getfem/>  
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